

• Energía de Deformación y Teorema de Castigliano

↳ Energía por est. axial: $U_{tr} = \int_0^L \frac{1}{2E} \frac{P^2}{A} dx$

↳ Energía por flexión: $U_{fl} = \int_0^L \frac{1}{2E} \frac{M(x)^2}{I_z} dx$

↳ Energía por torsión: $U_{tor} = \int_0^L \frac{1}{2G} \frac{T^2}{J} dx$

↳ Energía por corte: $U_{car} = \int_0^L \frac{3}{5GA} V(x)^2 dx$
(Sección rectangular)

Energía de deformación total U_T es la suma de las energías

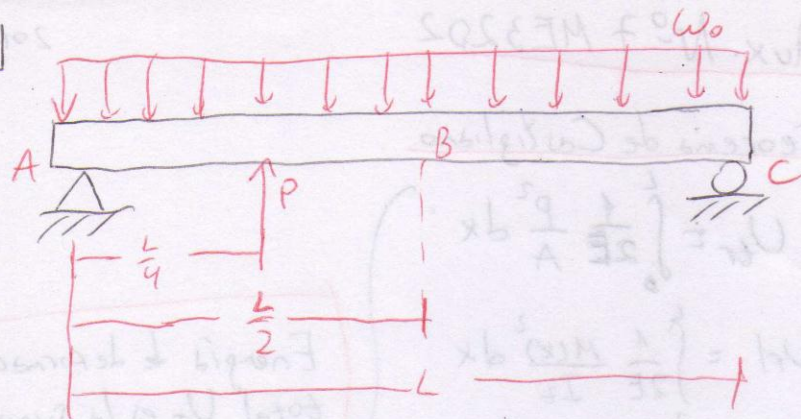
• Teorema de Castigliano

* Deflexión en el punto i : $\delta_i = \frac{\partial U_T}{\partial F_i} (F_i: \text{fz. aplicada en punto } i)$

* Ángulo de rotación en i : $\theta_i = \frac{\partial U_T}{\partial C_i} (C_i: \text{Momento aplicado en el punto } i)$

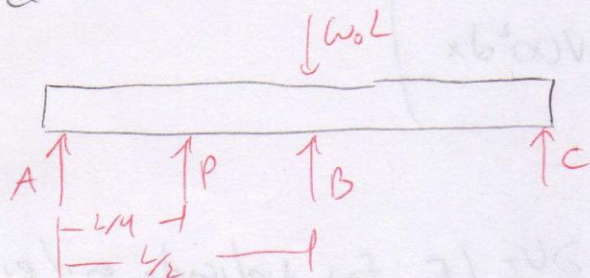
* Si no hay fz./momento aplicado en el punto de interés, se genera uno virtual, y luego de la derivación (Aplicación del Teorema de Castigliano) se hace igual a cero. δ

P1)



* En general se puede despreciar la energía por corte si $L \gg h$ (A/b). Si no se dice nada, es preferible calcularla.

• Para aplicar Castigliano, debemos "Inventar" una fuerza en B. Haciendo DCL:



$$\sum_A M_z = 0$$

$$\hookrightarrow \frac{PL}{4} + \frac{BL}{2} + CL - \frac{w_0 L^2}{2} = 0$$

$$\therefore C = \frac{w_0 L}{2} - \frac{P}{4} - \frac{B}{2}$$

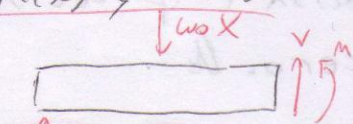
$$\sum F_y = 0 \Rightarrow A + P + B + C - w_0 L = 0 \Rightarrow A = w_0 L - P - B - C$$

$$= w_0 L - \frac{w_0 L^2}{2} + \frac{P}{4} + \frac{B}{2} - P - B$$

$$\therefore A = \frac{w_0 L}{2} - \frac{3P}{4} - \frac{B}{2}$$

• Calculando $M(x)$ y $V(x)$:

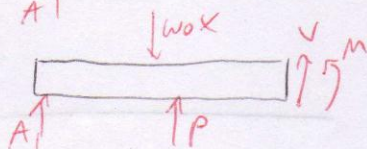
$$0 < x < \frac{L}{4}$$



$$V(x) = w_0 x - A$$

$$M(x) = Ax - w_0 \frac{x^2}{2}$$

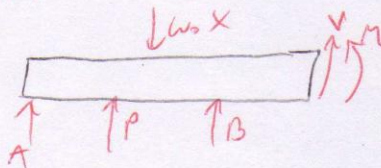
$$\frac{L}{4} < x < \frac{L}{2}$$



$$V(x) = w_0 x - A - P$$

$$M(x) = Ax - w_0 \frac{x^2}{2} + P(x - \frac{L}{4})$$

$$\frac{L}{2} < x < L$$



$$V(x) = w_0 x - A - P - B$$

$$M(x) = Ax - w_0 \frac{x^2}{2} + P(x - \frac{L}{4}) + B(x - \frac{L}{2})$$

- Calculando las energías por flexión y por corte, y aplicando Castiglione

$$U_{fl} = \frac{1}{2EI} \int_0^L M(x)^2 dx \Rightarrow \delta A = \frac{\partial U_{fl}}{\partial B} = \frac{1}{EI} \int_0^L M(x) \frac{\partial M(x)}{\partial B} dx$$

$$U_{cor} = \frac{3}{5GA} \int_0^L V(x)^2 dx \Rightarrow \delta_{cor} = \frac{\partial U_{cor}}{\partial B} = \frac{6}{5GA} \int_0^L V(x) \frac{\partial V(x)}{\partial B} dx$$

$$* M(x) = \begin{cases} Ax - \omega_0 \frac{x^2}{2} \\ Ax - \omega_0 \frac{x^2}{2} + P(x - \frac{L}{4}) \\ Ax - \omega_0 \frac{x^2}{2} + P(x - \frac{L}{4}) + B(x - \frac{L}{2}) \end{cases} \Rightarrow \frac{\partial M(x)}{\partial B} = \begin{cases} \frac{\partial A}{\partial B} x \\ \frac{\partial A}{\partial B} x \\ \frac{\partial A}{\partial B} x + (x - \frac{L}{2}) \end{cases}$$

$$* V(x) = \begin{cases} \omega_0 x - A \\ \omega_0 x - A - P \\ \omega_0 x - A - P - B \end{cases} \Rightarrow \frac{\partial V(x)}{\partial B} = \begin{cases} -\frac{\partial A}{\partial B} \\ -\frac{\partial A}{\partial B} \\ -\frac{\partial A}{\partial B} - 1 \end{cases}$$

$$\Rightarrow \text{Dado que } A = \frac{\omega_0 L}{2} - \frac{3P}{4} - \frac{B}{2} \Rightarrow \frac{\partial A}{\partial B} = -\frac{1}{2}$$

$$\Rightarrow \delta_{fl} = \frac{1}{EI} \left[\int_0^{L/4} \left(-\frac{x}{2}\right) (Ax - \omega_0 \frac{x^2}{2}) dx + \int_{L/4}^{L/2} \left(-\frac{x}{2}\right) (Ax - \omega_0 \frac{x^2}{2} + P(x - \frac{L}{4})) dx + \int_{L/2}^L \left(-\frac{x}{2} + (x - \frac{L}{2})\right) (Ax - \omega_0 \frac{x^2}{2} + P(x - \frac{L}{4}) + B(x - \frac{L}{2})) dx \right]$$

- Calculando las integrales, evaluándolas y simplificando (y haciendo $B=0$)

$$\delta_{fl} = \frac{1}{EI} \left(-\frac{5}{384} \omega_0 L^4 + \frac{11}{768} PL^3 \right)$$

(Detalle de cálculo en apéndice A.1)

- Por otro lado

$$\Rightarrow \delta_{cor} = \frac{6}{5GA} \left[\int_0^{L/4} \frac{1}{2} (\omega_0 x - A) dx + \int_{L/4}^{L/2} \frac{1}{2} (\omega_0 x - A - P) dx + \int_{L/2}^L \left(-\frac{1}{2}\right) (\omega_0 x - A - P - B) dx \right]$$

$$= \frac{6}{5GA} \left(-\frac{\omega_0 L^2}{4} - \frac{BL}{4} + \frac{PL}{8} \right)$$

$$\delta_{cor} = \frac{6}{5GA} \left(-\frac{\omega_0 L^2}{4} + \frac{PL}{8} \right)$$

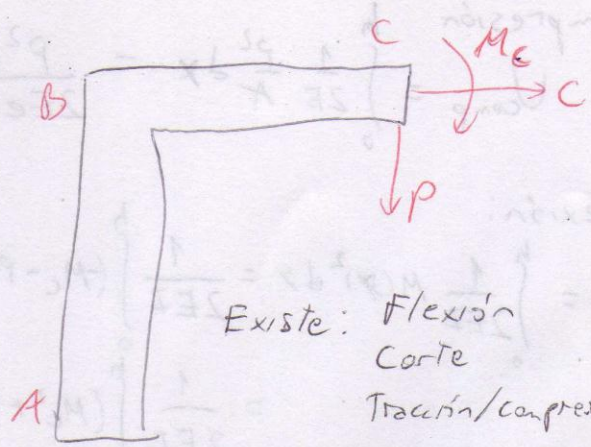
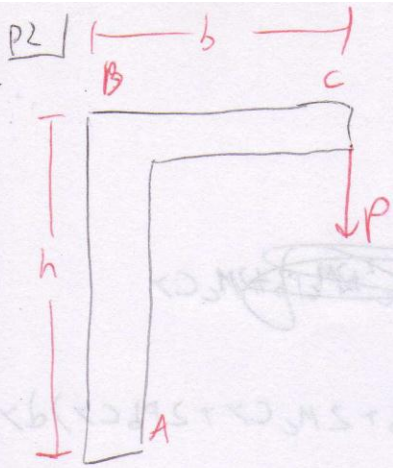
• Finalmente, dado que la deflexión en ese punto debe ser cero, se tiene que

$$\delta_{fl} + \delta_{cor} = \delta_T = 0$$

$$\frac{1}{EI} \left(-\frac{5}{384} \omega_0 L^4 + \frac{11}{768} PL^3 \right) + \frac{6}{5GA} \left(-\frac{\omega_0 L^2}{4} + \frac{PL}{8} \right) = 0$$

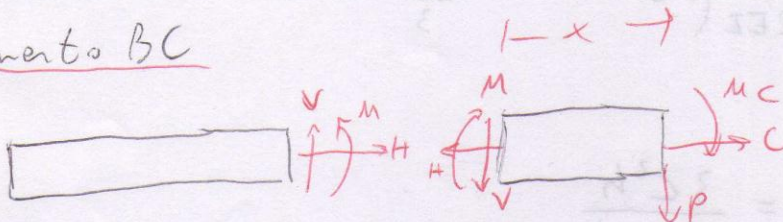
$$P \left(\frac{11}{768} \frac{L^3}{EI} + \frac{3}{20} \frac{L}{GA} \right) = \left(\frac{5}{384} \frac{L^4}{EI} + \frac{3}{10} \frac{L^2}{GA} \right) \omega_0$$

$$P = 2 \left(\frac{25 L^3 GA + 576 L EI}{55 L^2 GA + 576 EI} \right) \omega_0$$



Existe: Flexión
Corte
Tracción/compresión

Segmento BC



$$\begin{aligned} H &= C \\ V &= -P \\ M &= -Px - M_c \end{aligned}$$

→ Tracción

$$U_{tr} = \int_0^b \frac{1}{2E} \frac{C^2}{A} dx = \frac{C^2 b}{2EA} = \frac{C^2 b}{2Eed}$$

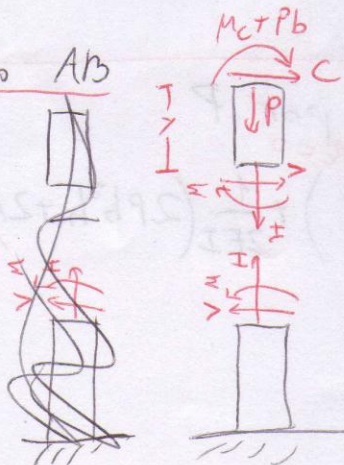
→ Flexión

$$U_H = \int_0^b \frac{M(x)^2}{2EI} dx = \frac{1}{2EI} \int_0^b (-Px - M_c)^2 dx = \frac{1}{2EI} \left[\frac{P^2 b^3}{3} + M_c P b^2 + M_c^2 b \right]$$

→ Corte

$$U_{cor} = \int_0^b \frac{3}{56A} V(x)^2 dx = \frac{3P^2 b}{56A} = \frac{3P^2 b}{56ed}$$

Segmento AB



$$\begin{aligned} H &= -P \\ V &= -C \\ M &= -M_c - Pb - Cy \end{aligned}$$

→ Compresión

$$U_{\text{comp}} = \int_0^h \frac{1}{2E} \frac{P^2}{A} dx = \frac{P^2 h}{2Eed}$$

→ Flexión:

$$U_{\text{fl}} = \int_0^h \frac{1}{2EI} M(x)^2 dx = \frac{1}{2EI} \int_0^h (-M_c - Pb - Cx)^2 dx$$

~~$(M_c^2 + 2M_c Pb + 2M_c Cx + P^2 b^2 + 2Pb Cx + C^2 x^2)$~~

$$= \frac{1}{2EI} \int_0^h (M_c^2 + P^2 b^2 + C^2 x^2 + 2M_c Pb + 2M_c Cx + 2Pb Cx) dx$$

$$= \frac{1}{2EI} \left(M_c^2 h + P^2 b^2 h + \frac{C^2 h^3}{3} + 2M_c Pb h + M_c C h^2 + Pb C h^2 \right)$$

→ Corte:

$$U_{\text{cor}} = \frac{3}{5GA} \int_0^h V(x)^2 dx = \frac{3C^2 h}{5Ged}$$

• Luego, la energía total de deformación está dada por

$$U_T = U_{\text{tr}} + U_{\text{comp}} + U_{\text{fl}}^{AB} + U_{\text{fl}}^{BC} + U_{\text{cor}}^{AB} + U_{\text{cor}}^{BC}$$

$$U_T = \frac{C^2 b}{2Eed} + \frac{P^2 h}{2Eed} + \frac{1}{2EI} \left(\frac{P^2 b^3}{3} + M_c P b^2 + M_c^2 b \right)$$

$$+ \frac{1}{2EI} \left(M_c^2 h + P^2 b^2 h + \frac{C^2 h^3}{3} + 2M_c P b h + M_c C h^2 + P b C h^2 \right)$$

$$+ \frac{3P^2 b}{5Ged} + \frac{3C^2 h}{5Ged}$$

• Deflexión vertical δ_y en C → producida por P

$$\left(\frac{\partial U_T}{\partial P} \right)_{\substack{M_c=0 \\ C=0}} = \frac{Ph}{Eed} + \frac{1}{2EI} \left(\frac{2}{3} P b^3 + \cancel{M_c b^2} \right) + \frac{1}{2EI} \left(2P b^2 h + \cancel{2M_c b h} + \cancel{b C h^2} \right)$$

$$+ \frac{6Pb}{5Ged}$$

$M_c=0 \quad C=0$

$$= \frac{Ph}{Eed} + \frac{Pb^3}{3EI} + \frac{Pb^2h}{EI} + \frac{6Pb}{5Ged} \quad / \quad I_z = \frac{ed^3}{12}$$

$$\therefore \left(\frac{\partial U_T}{\partial P} \right)_{\substack{M_c=0 \\ C=0}} = \delta_y = \frac{P}{Eed} \left(h + \frac{4b^3}{d^2} + \frac{12b^2h}{d^2} \right) + \frac{6Pb}{5Ged}$$

$$\therefore \delta_y = \frac{4P}{Eed} \left(\frac{h}{4} + \frac{b^3 + 3b^2h}{d^2} \right) + \frac{6}{5} \frac{Pb}{Ged}$$

• Deflexión horizontal δ_x en C $\rightarrow$ Producida por C

$$\left(\frac{\partial U_T}{\partial C} \right)_{\substack{M_c=0 \\ C=0}} = \frac{Cb}{Eed} + \frac{Ch^3}{3EI} + \frac{M_ch^2}{2EI} + \frac{Pbh^2}{2EI} + \frac{6Ch}{5Ged}$$

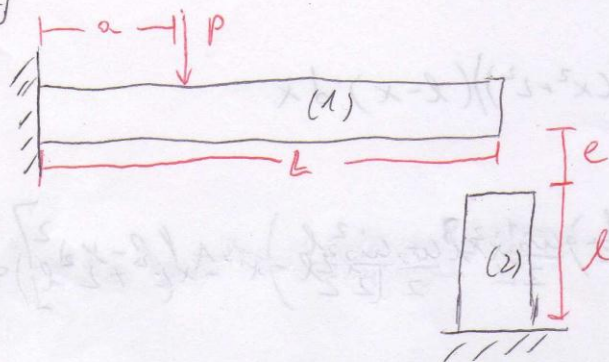
$$\therefore \delta_x = \frac{6Pbh^2}{Eed^3}$$

• ~~Deflexión~~ Ángulo de rotación en C $\rightarrow$ Producido por M_c

$$\left(\frac{\partial U_T}{\partial M_c} \right)_{\substack{M_c=0 \\ C=0}} = \frac{Pb^2}{2EI} + \frac{M_cb}{EI} + \frac{M_ch}{EI} + \frac{Pbh}{EI} + \frac{Ch^2}{2EI}$$

$$\Rightarrow \theta_c = \frac{Pb^2}{2EI} + \frac{Pbh}{EI} \Rightarrow \theta_c = \frac{6Pb}{Eed^3} (b + 2h)$$

P3/



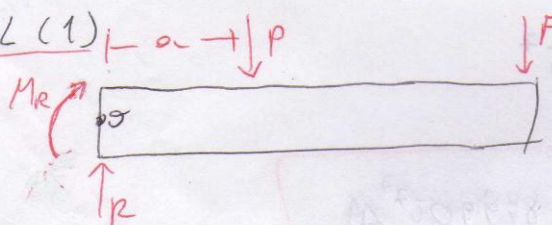
E_1, I_2, AE_2, A_2
(Considerar sólo energía por flexión)

i) Dado que nos piden deflexión y NO hay una fuerza en el punto solicitado, inventaremos una y luego la anularemos:

ii) Dos casos: Barra (1) toca o no toca a (2)

* Barra (1) no toca a (2)

• DCL (1)



$$\sum F_y = 0$$

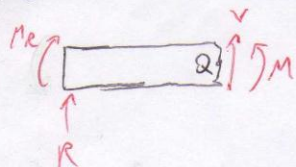
$$R = P + F$$

$$\sum M_z = 0$$

$$M_R = -(a \cdot P + L \cdot F)$$

• Cálculo de fuerzas internas $M(x)$:

$$0 < x < a$$



$$\sum M_z = 0 \Rightarrow M(x) = M_R + Rx$$

$$M(x) = -aP - LF + Rx + Fx$$

$$M(x) = F(x - L) + P(x - a)$$

• Teorema de Castiglione:

$$\delta_{FI} = \frac{\partial U_{FI}}{\partial F} = \frac{1}{2EI} \int_0^L 2M(x) \cdot \frac{dM(x)}{dF} dx = \frac{1}{EI} \int_0^L M(x) \frac{dM(x)}{dF} dx$$

$$* \frac{dM(x)}{dF} = \begin{cases} (x-L) & 0 < x < a \\ (x-L) & a < x < L \end{cases}$$

$$\begin{aligned} \Rightarrow \delta_{FI} &= \frac{1}{EI} \left(\int_0^a (F(x-L) + P(x-a))(x-L) dx + \int_a^L F(x-L)^2 dx \right) \\ &= \frac{1}{EI} \left(\int_0^a F(x^2 - 2Lx + L^2) + P(x^2 - x(a+L) + aL) dx + \int_a^L F(x^2 - 2Lx + L^2) dx \right) \\ &= \frac{1}{EI} \left(F \left(\frac{a^3}{3} - La^2 + L^2a \right) + P \left(\frac{a^3}{3} - \frac{a^2}{2}(a+L) + aL \right) + (F \cdot (\dots)) \right) \Big|_a^L \end{aligned}$$

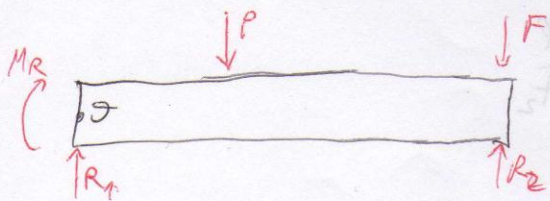
* Dado que $F=0$

$$\delta_{FI} = P \left(\frac{a^3}{3} - \frac{a^3}{2} - \frac{a^2L}{2} + a^2L \right) \cdot \frac{1}{EI}$$

$$\delta_{FI} = \frac{P}{6EI} (3L - a) a^2$$

* Barra (1) si toca a (2): Aplicamos teo. de Castiglione en ambos casos

• DCL (1)



$$\sum F_y = 0$$

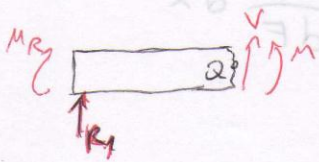
$$R_1 = P + F - R_2$$

$$\sum M_z = 0$$

$$M_R = -a \cdot P - FL + R_2 L$$

• Cálculo de $M(x)$

↳ $0 < x < a$

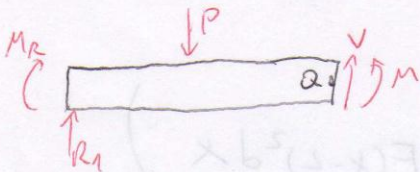


$$\sum \mathcal{M}_z = 0 \Rightarrow M(x) = M_R + R_1 x$$

$$= -aP - FL + R_2 L + Px + Fx - R_2 x$$

$$M(x) = P(x-a) + (F-R_2)(x-L)$$

↳ $a < x < L$



$$\sum \mathcal{M}_z = 0 \Rightarrow M(x) = M_R + R_1 x - P(x-a)$$

$$M(x) = (F-R_2)(x-L)$$

• Castigliano

$$\ast \frac{dM(x)}{dF} = x-L, \forall x \in [0, L]$$

$$\Rightarrow \delta F l = \frac{1}{EI} \int_0^a (P(x-a) + (F-R_2)(x-L))(x-L) dx + \int_a^L (F-R_2)(x-L)^2 dx$$

* $F \neq 0$

$$= \frac{1}{EI} \left(P \left(\frac{a^3}{3} - \frac{a^2}{2}(a+L) + a^2 L \right) + R_2 \frac{(a-L)^3}{3} - \frac{R_2 a^3}{3} - R_2 \frac{(a-L)^3}{3} \right)$$

$$\delta F l = \frac{a^2 L P}{2} - \frac{a^3 P}{6} - \frac{R_2 a^3}{3}$$

• DCL (2)

* Energía por compresión: $U_T = \frac{R_2^2 l}{2 E_2 A_2}$



Castigliano:

$$\Rightarrow \delta_T = \frac{\partial U_T}{\partial R_2} = \frac{R_2 l}{E_2 A_2}$$

* Vigas (1) se deflecta e más la que se dobla (2):

$$\delta_{fl} = e + \delta_T$$

$$\frac{\omega^2 PL}{2} - \frac{\omega^3 P}{6} - \frac{R_2 \omega^3}{3} = e + \frac{R_2 l}{E_2 A_2}$$

$$\Rightarrow R_2 \left(\frac{\omega^3}{3} + \frac{l}{E_2 A_2} \right) = \frac{P \omega^2}{6} (3L - e) + e$$

$$\therefore R_2 = \frac{P \omega^2 E_2 A_2 (3L - e)}{3l + \omega^3 E_2 A_2} + \left(\frac{e}{\frac{\omega^3}{3} + \frac{l}{E_2 A_2}} \right)$$