Pauta Aux. N°6 ME3202 2014

• Cálculo de deflexión por integración:

$$\frac{d^4 y}{dx^4} = -\frac{w(x)}{EI}$$ ← Fza distribuida

* Modelación de fzas mediante función escalón ($r(x-x_0)$) para fzas distribuidas, y delta de Dirac ($\delta(x-x_0)$) para fzas puntuales

* Lo que ocurre en los extremos de la viga se reemplaza por condiciones de borde (Son necesarias 4 cond. de borde)

* Las cond. de borde son condiciones en la deflexión en los extremos, el ángulo de deflexión ($\frac{dy}{dx}$), el momento ($\frac{d^2y}{dx^2}$) ó ~~de~~ Fza de corte ($\frac{d^3y}{dx^3}$)

$$\hookrightarrow \boxed{M = EI\frac{d^2y}{dx^2}} \wedge \boxed{V = -EI\frac{d^3y}{dx^3}}$$

* Algunos bordes:

$\Rightarrow$ No hay deflexión $\rightarrow y(0) = 0$
No hay momento $\rightarrow \frac{d^2y}{dx^2} = 0$

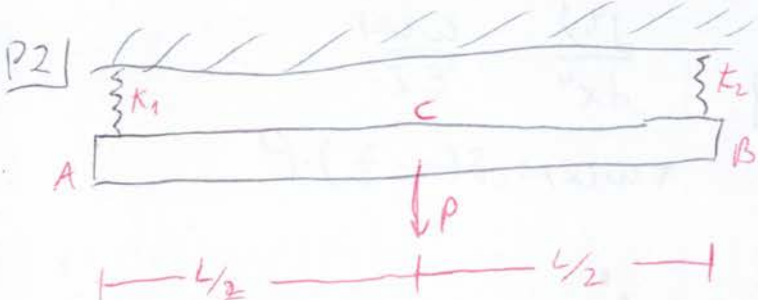
$\Rightarrow$ No hay deflexión $\rightarrow y(0) = 0$
No hay ángulo $\rightarrow \frac{dy}{dx}(0) = 0$

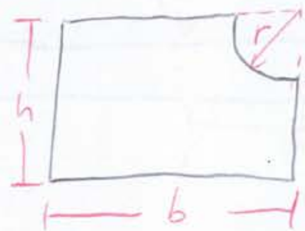
$\Rightarrow$ No hay momento $\rightarrow \frac{d^2y}{dx^2}(0) = 0$

Balance de fzas $\rightarrow V(0) = -F = -k\,y(0)$

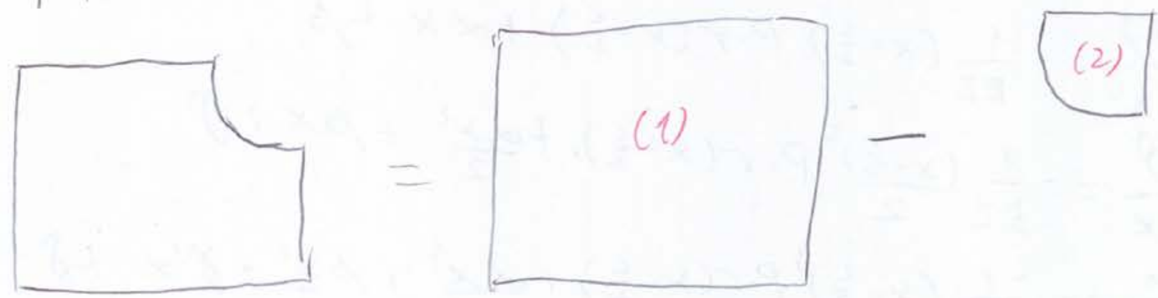
$$\therefore -EI\frac{d^3y}{dx^3}(0) = -k\,y(0)$$

P2)

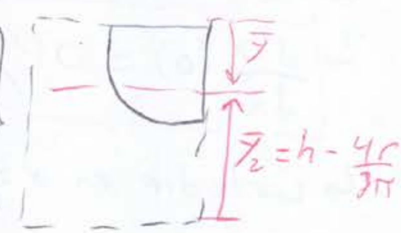




a) Propiedades de área:



(1): $\bar{y}_1 = \frac{h}{2}$ ∧ $I_1 = \frac{b\cdot h^3}{12}$

(2):  ~~$\bar{y} = \frac{4r}{3\pi}$~~ ∧ $I_2 = \frac{\pi r^4}{16}$ 

$$\Rightarrow \bar{y}_T = \frac{\bar{y}_1 A_1 - \bar{y}_2 A_2}{A_1 - A_2} = \frac{\frac{h}{2}\cdot hb - \overbrace{\left(h - \frac{4r}{3\pi}\right)}^{\text{Desde la base}}\cdot \frac{1}{4}\pi r^2}{hb - \frac{1}{4}\pi r^2} \cancel{= 4{,}567\ cm}$$

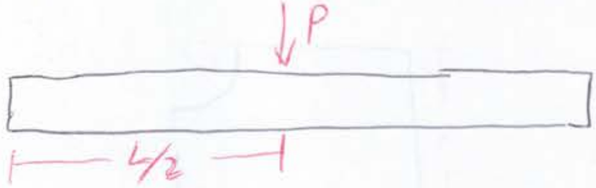
$$\boxed{\bar{y}_T = 4{,}567\ cm}$$

$$\Rightarrow \bar{I}_1 = I_1 + \delta^2 A = \frac{bh^3}{12} + \left(\frac{h}{2} - \bar{y}_T\right)^2 bh \Rightarrow \boxed{\bar{I}_1 = 1{,}278\cdot 10^{-5}\ m^4}$$

$$\bar{I}_2 = I_2 + \delta^2 A = \frac{\pi r^4}{16} + \left(\bar{y}_T - \left(h + \frac{4r}{3\pi}\right)\right)^2 \cdot \frac{1}{4}\pi r^2 \Rightarrow \boxed{\bar{I}_2 = 3{,}38\cdot 10^{-6}\ m^4}$$

$$\therefore \bar{I}_T = \bar{I}_1 - \bar{I}_2 \Longrightarrow \boxed{\bar{I}_T = 9{,}401\cdot 10^{-6}\ m^4}$$

b) Aplicando la ec. de la curva elástica



$$\frac{d^4 y}{dx^4} = -\frac{\omega(x)}{EI}$$

$$* \ \omega(x) = \delta(x - \tfrac{L}{2}) \cdot P$$

· Integrando:

$$\rightsquigarrow \frac{d^3 y}{dx^3} = -\frac{1}{EI} \cdot P \cdot r(x - \tfrac{L}{2}) + \alpha$$

$$\rightarrow \frac{d^2 y}{dx^2} = -\frac{1}{EI} (x - \tfrac{L}{2}) \cdot P \cdot r(x - \tfrac{L}{2}) + \alpha x + \beta$$

$$\rightsquigarrow \frac{dy}{dx} = -\frac{1}{EI} \frac{(x - \frac{L}{2})^2}{2} P \cdot r(x - \tfrac{L}{2}) + \frac{\alpha x^2}{2} + \beta x + \gamma$$

$$\Rightarrow \hat{y}(x) = \frac{-1}{6EI} (x - \tfrac{L}{2})^3 P r(x - \tfrac{L}{2}) + \frac{\alpha x^3}{6} + \beta \frac{x^2}{2} + \gamma x + \delta$$

· Condiciones de Borde:

$$\boxed{\frac{d^2 y}{dx^2}(0) = 0 \ (1)} \Big/ \boxed{\frac{d^2 y}{dx^2}(L) = 0 \ (2)}$$

↳ Corte dif. en $x = 0$ ⇝

$$\left.\begin{matrix} V(0) + A = 0 \\ \Rightarrow V(0) = -A \end{matrix}\right.$$

$$* \ V(x) = -EI \frac{d^3 y}{dx^3}(x) \quad y \quad A = -k_1 \cdot \hat{y}(0)$$

$$\boxed{\therefore -EI \frac{d^3 y}{dx^3}(0) = k_1 \hat{y}(0) \ (3)}$$

↳ Corte dif. en $x = L$

$$\left.\begin{matrix} V(L) - B = 0 \Rightarrow V(L) = B \\ * \ B = -k_2 \hat{y}(L) \end{matrix}\right.$$

$$\boxed{\therefore EI \frac{d^3 y}{dx^3}(L) = k_2 \hat{y}(L) \ (4)}$$

· Aplicando los cond. de Borde

(1) $\frac{d^2y}{dx^2}(0) = 0 \Rightarrow \boxed{\beta = 0}$

(2) $\frac{d^2y}{dx^2}(L) = 0 \sim -\frac{1}{EI}(L-\frac{L}{2})\cdot P + \alpha L = 0$

$\Rightarrow \boxed{\alpha = \frac{P}{2EI}}$

(3) $-EI\frac{d^3y}{dx^3}(0) = k_1 y(0)$

$-\alpha EI = k_1\delta \sim \delta = -\frac{\alpha EI}{k_1} = -\frac{P}{2\cancel{EI}}\cdot\frac{\cancel{EI}}{k_1} \Rightarrow \boxed{\delta = \frac{-P}{2k_1}}$

(4) $EI\frac{d^3y}{dx^3}(L) = k_2 y(L)$

$-P + \alpha EI = k_2\left(-\frac{1}{6EI}(L-\frac{L}{2})^3 P + \frac{\alpha L^3}{6} + \gamma L + \delta\right)$

~~Despejando~~

$-P + \frac{P}{2} = k_2\left(-\frac{L^3 P}{48EI} + \frac{\overset{\frac{P}{2EI}}{\alpha} L^3}{6} - \frac{P}{2k_1}\right) + k_2 L\gamma$

$$\boxed{\therefore \gamma = \frac{1}{k_2 L}\left(-\frac{P}{2} - k_2 P\left(-\frac{L^3}{48EI} + \frac{L^3}{12EI} - \frac{1}{2k_1}\right)\right)}$$

· Una vez calculadas todas las constantes, evaluemos $y(\frac{L}{2})$ para encontrar el desplazamiento en el punto C.

$y(\frac{L}{2}) = \cancel{\alpha\frac{L^3}{6} + \beta\frac{L^2}{2} + \gamma L + \delta} = \frac{\alpha(\frac{L}{2})^3}{6} + \gamma\frac{L}{2} + \delta$

· Calculando las ctes: $\alpha = 1{,}4\cdot 10^{-3}$

$\gamma = -7{,}4\cdot 10^{-3}$

$\delta = -8{,}3\cdot 10^{-3}$

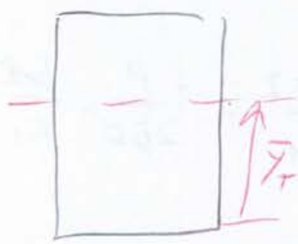
$$\boxed{\therefore y(\tfrac{L}{2}) = -0{,}0241\ m = -2{,}41\ cm}$$

~~a)~~ $\cancel{M = EI \frac{d^2y}{dx^2} \Rightarrow M_{max}}$ ~~~ derivamos $M(x)$, obtenemos x para luego evaluar en $M(x)$~~

$$\cancel{\Rightarrow \frac{d^3y}{dx^3} = 0 \rightsquigarrow -\frac{1}{EI} P \cdot r\left(x - \frac{L}{2}\right) + \frac{P}{2EI}}$$

c) $M_{max} \sim \left(\frac{d^2y}{dx^2}\right)_{max} \sim x = \frac{L}{2}$

$Y_{max} \sim$



$* \; y = -\overline{y}_T = -4{,}867 \text{ cm}$

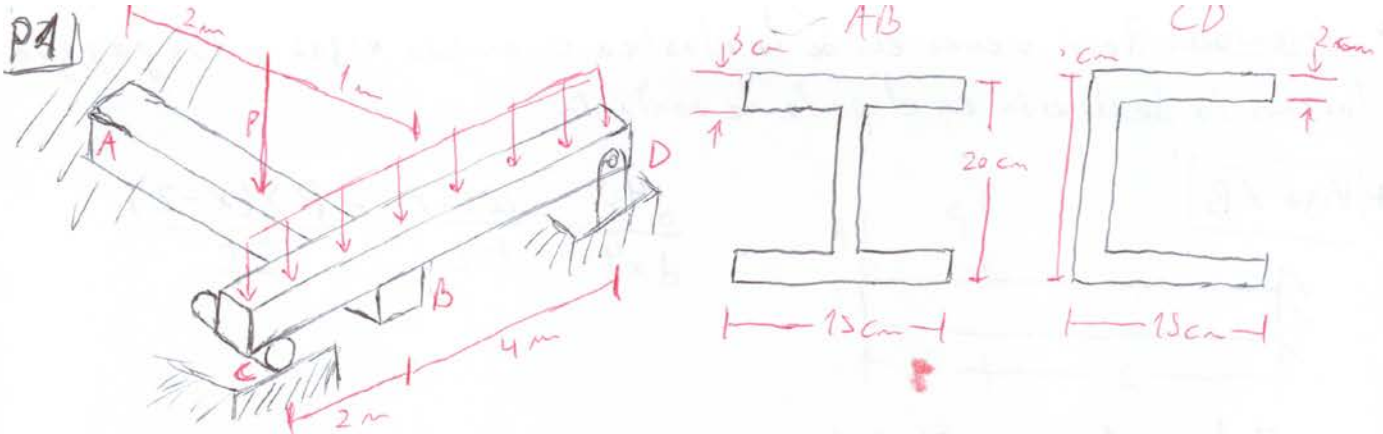
$* \; y = h - \overline{y}_T = 5{,}433 \text{ cm}$ → Máximo (Extr. sup)

$$\therefore \sigma_{max} = -\frac{M}{I} y \quad \wedge \quad M = EI \frac{d^2y}{dx^2}$$

$$\Rightarrow \sigma_{max} = -E y_{max} \left(\frac{d^2y}{dx^2}\right)_{max} \Bigg\} \; \frac{d^2y}{dx^2}(L/2) = \frac{PL}{4EI}$$

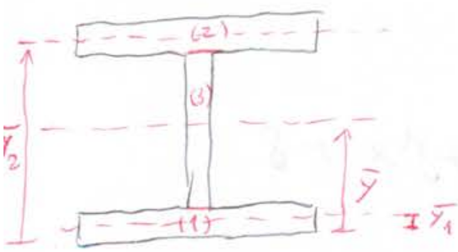
$$\boxed{\therefore \sigma_{max} = -43{,}342 \text{ MPa}}$$

P1)



• Cálculo de propiedades de área: $\left(\bar{I} = I + \delta^2 \cdot A\right)$

* Viga AB

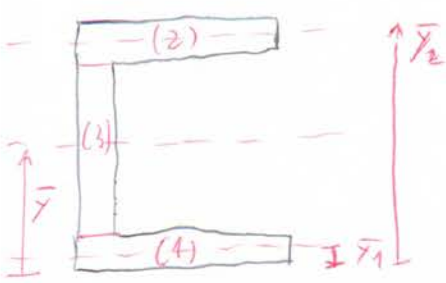


$\bar{y} = 0{,}1\ m$; $\bar{y}_1 = 0{,}015\ m$; $\bar{y}_2 = 0{,}185$

$$\bar{I}_z = \bar{I}_1 + \bar{I}_2 + \bar{I}_3 = \frac{0{,}15 \cdot 0{,}03^3}{12} + (\bar{y} - \bar{y}_1)^2 \cdot 0{,}03 \cdot 0{,}15 + \frac{0{,}15 \cdot 0{,}03^3}{12} + (\bar{y}_2 - \bar{y})^2 \cdot 0{,}03 \cdot 0{,}15 + \frac{0{,}03 \cdot 0{,}14^3}{12}$$

$$\boxed{\bar{I}_z^{AB} = 7{,}256 \cdot 10^{-5}\ m^4}$$

* Viga CD



$\bar{y} = 0{,}075\ m$; $\bar{y}_1 = 0{,}01\ m$; $\bar{y}_2 = 0{,}14\ m$

$$\rightsquigarrow \bar{I}_1 = \frac{0{,}15 \cdot 0{,}02^3}{12} + (0{,}075 - 0{,}01)^2 \cdot 0{,}15 \cdot 0{,}02$$

$$\boxed{\bar{I}_1 = 1{,}278 \cdot 10^{-5}}$$

$$\rightsquigarrow \bar{I}_2 = \frac{0{,}15 \cdot 0{,}02^3}{12} + (0{,}14 - 0{,}075)^2 \cdot 0{,}15 \cdot 0{,}02$$

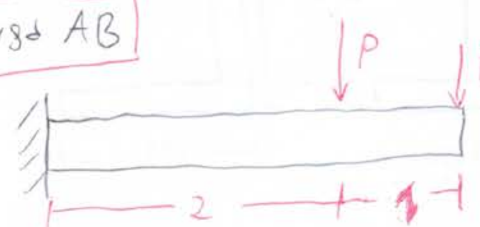
$$\boxed{\bar{I}_2 = 1{,}278 \cdot 10^{-5}}$$

$$\rightsquigarrow \bar{I}_3 = \frac{0{,}02 \cdot 0{,}11^3}{12} = 2{,}218 \cdot 10^{-6}$$

$$\boxed{\therefore\ \bar{I}_z^{CD} = 2{,}778 \cdot 10^{-5}\ m^4}$$

• Deflexión: Resolveremos ec. de la elástica en ambas vigas y luego igualaremos la deflexión en el punto de contacto

+ $\boxed{\text{Viga AB}}$



$$\frac{d^4y}{dx^4} = -\frac{\omega(x)}{EI} = -\frac{P\cdot\delta(x-2)}{EI}$$

$$\rightsquigarrow \frac{d^3y}{dx^3} = -\frac{1}{EI} P\, r(x-2) + \alpha$$

$$\rightsquigarrow \frac{d^2y}{dx^2} = -\frac{1}{EI}(x-2)P\cdot r(x-2) + \alpha x + \beta$$

$$\rightsquigarrow \frac{dy}{dx} = -\frac{1}{2EI}(x-2)^2 P\, r(x-2) + \alpha\frac{x^2}{2} + \beta x + \gamma$$

$$\rightsquigarrow y(x) = -\frac{1}{6EI}(x-2)^3 P\, r(x-2) + \alpha\frac{x^3}{6} + \beta\frac{x^2}{2} + \gamma x + \delta$$

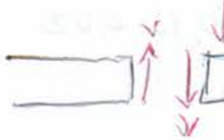
• Condiciones de Borde

$$* \ y(0) = 0 \Rightarrow \boxed{\delta = 0}$$

$$* \ \frac{dy}{dx}(0) = 0 \Rightarrow \boxed{\gamma = 0}$$

$$* \ \frac{d^2y}{dx^2}(3) = 0 \Rightarrow -\frac{1}{EI}(3-2)\cdot P + 3\alpha + \beta = 0$$

$$\boxed{\beta = \frac{1}{EI}\cdot P - 3\alpha}$$

$*$ 

$$\Rightarrow EI\frac{d^3y}{dx^3}(3) = R$$

$$-P + \alpha EI = R \Rightarrow \boxed{\alpha = \frac{1}{EI}(R+P)}$$

$$\boxed{\beta = \frac{-1}{EI}(3R+2P)}$$

$$\therefore y(x) = -\frac{1}{EI}\Big((x-2)^3 P\, r(x-2) - (R+P)\, x^3 + (3R+2P) \ldots$$

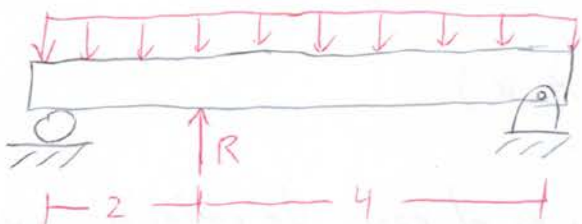
• La deflexión en el extremo de la viga:

$$y(3) = \frac{1}{EI}\left(-\frac{1}{6}P + \frac{9}{2}R + \frac{9}{2}P - \frac{27}{2}R - 9P\right)$$

$$= \frac{1}{EI}\left[\left(-\frac{1}{6} + \frac{9}{2} - 9\right)P + \left(\frac{9}{2} - \frac{27}{2}\right)R\right]$$

$$\boxed{y(3) = \frac{1}{EI}\left(-4{,}667P - 9R\right)} \quad (*)$$

* $\boxed{\text{Viga CD}}$



$$\frac{d^4 y}{dx^4} = -\frac{1}{EI}\left(\omega_0 - R\delta(x-2)\right)$$

$$\rightsquigarrow \frac{d^3 y}{dx^3} = -\frac{1}{EI}\left(\omega_0 x - R r(x-2)\right) + \alpha$$

$$\rightsquigarrow \frac{d^2 y}{dx^2} = -\frac{1}{EI}\left(\omega_0 \frac{x^2}{2} - (x-2)R r(x-2)\right) + \alpha x + \beta$$

$$\rightsquigarrow \frac{dy}{dx^2} = -\frac{1}{EI}\left(\omega_0 \frac{x^3}{6} - \frac{(x-2)^2}{2} R r(x-2)\right) + \alpha \frac{x^2}{2} + \beta x + \gamma$$

$$\rightsquigarrow y(x) = -\frac{1}{EI}\left(\omega_0 \frac{x^4}{24} - \frac{(x-2)^3}{6} R r(x-2)\right) + \alpha \frac{x^3}{6} + \beta \frac{x^2}{2} + \gamma x + \delta$$

• Condiciones de Borde

* $y(0) = 0 \Rightarrow \boxed{\delta = 0}$

* $\frac{d^2 y}{dx^2}(0) = 0 \Rightarrow \boxed{\beta = 0}$

* $\frac{d^2 y}{dx^2}(6) = 0 \Rightarrow -\frac{1}{EI}\left(\omega_0 \frac{6^2}{2} - (6-2)R\right) + 6\alpha = 0$

$$\boxed{\alpha = \frac{1}{6EI}\left(18\omega_0 - 4R\right)}$$

* $y(6) = 0 \Rightarrow -\frac{1}{EI}\left(\omega_0 \cdot \frac{6^4}{24} - \frac{(6-2)^3}{6} R\right) + \frac{1}{6EI}(18\omega_0 - 4R) \cdot \frac{6^3}{6} \ldots$

$$\boxed{y_1 = \frac{1}{EI}\left(-9w_0 + \frac{20}{9}R\right)}$$

$$\Rightarrow y(x) = -\frac{1}{EI}\left(w_0\frac{x^4}{24} - \frac{(x-2)^3}{6}R \cdot r(x-2) - \frac{x^3}{6}\left(3w_0 - \frac{2}{3}R\right) - x\left(-9w_0 + \frac{20}{9}R\right)\right)$$

- La deflexión en $x = 2$ m:

$$y(2) = -\frac{1}{EI}\left(w_0\frac{2^4}{24} - \frac{2^3}{6}\left(3w_0 - \frac{2}{3}R\right) - 2\left(-9w_0 + \frac{20}{9}R\right)\right)$$

$$= \frac{1}{EI}\left(-\frac{2}{3}w_0 + 4w_0 - \frac{8}{9}R - 18w_0 + \frac{40}{9}R\right)$$

$$\boxed{y(2) = \frac{1}{EI}\left(-\frac{44}{3}w_0 + \frac{32}{9}R\right)} \quad (**)$$

- Dado que las deflexiones son iguales en el punto de contacto, igualamos $(*)$ con $(**)$

$$\frac{1}{EI_{AB}}\left(-4{,}667P - 9R\right) = \frac{1}{EI_{CD}}\left(-14{,}667w_0 + 3{,}556R\right)$$

$$R\left(\frac{3{,}556}{I_{CD}} + \frac{9}{I_{AB}}\right) = \frac{14{,}667w_0}{I_{CD}} - \frac{4{,}667P}{I_{AB}}$$

$$\boxed{\therefore R = 792{,}195 \text{ N}}$$