(1)

(a) $g_r = \dfrac{\partial \vec{r}}{\partial r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

$|g_r| = 1$

$$\boxed{\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}}$$

$g_\theta = -r\sin\theta\,\hat{i} + r\cos\theta\,\hat{j}$

$|g_\theta| = r$

$$\boxed{\hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}}$$

(b) $\dot{\hat{e}}_r = -\sin\theta\cdot\dot{\theta}\,\hat{i} + \cos\theta\cdot\dot{\theta}\,\hat{j}$

$$\boxed{\dot{\hat{e}}_r = \hat{e}_\theta\cdot\dot{\theta}}$$

$\dot{\hat{e}}_\theta = -\cos\theta\,\dot{\theta}\,\hat{i} - \sin\theta\,\dot{\theta}\,\hat{j}$

$$\boxed{\dot{\hat{e}}_\theta = -\hat{e}_r\,\dot{\theta}}$$

$\dot{\hat{e}}_r \perp \hat{e}_r$

$\dot{\hat{e}}_\theta \perp \hat{e}_\theta$

②

ⓐ $\vec{R} = r\,\hat{r}$

$$\boxed{-\frac{GmM}{r^2}\hat{r} = m\frac{d^2}{dt^2}\vec{R}}$$

$$\boxed{-\frac{GM}{r^2}\hat{r} = \vec{a}}$$

ⓑ $\frac{d}{dt}(\vec{R}) = \frac{d}{dt}\left(r\hat{r}\right) = \dot{r}\hat{r} + r\dot{\hat{r}}$

$$= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$\frac{d^2}{dt^2}(\vec{R}) = \ddot{r}\hat{r} + \dot{r}\dot{\hat{r}} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\dot{\hat{\theta}}$$

$$\vec{a} = \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r}$$

$$a_r = \ddot{r} - r\dot{\theta}^2$$

$$a_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta}$$

ⓒ en $\hat{r}$ $\quad F = ma$

$$-\frac{GM}{r^2} = \ddot{r} - r\dot{\theta}^2 \qquad (1)$$

en $\hat{\theta}$

Se conserva el momentum angular.

$$0 = 2m\dot{r}\dot{\theta} + r\ddot{\theta}m$$

$$0 = \frac{d}{dt}(mr^2\dot{\theta}) \Rightarrow \boxed{L = mr^2\dot{\theta}} \qquad (2)$$

(d) $\frac{L}{mr^2} = \dot{\theta} \Rightarrow \frac{L^2}{m^2 r^4} = \dot{\theta}^2$

$$\Rightarrow -\frac{GM}{r^2} = \ddot{r} - \frac{L^2}{m^2 r^3}$$

$$\frac{d^2 r}{dt^2} = -\frac{GM}{r^2} + \frac{L^2}{m^2 r^3} \quad (c)$$

$$r = u^{-1}(\theta(t))$$

$$\frac{dr}{dt} = -u^{-2}\frac{du}{d\theta}\cdot\frac{d\theta}{dt} = -u^{-2}\frac{du}{d\theta}\cdot\frac{L}{m}u^2 = -\frac{du}{d\theta}\cdot\frac{L}{m}$$

$$\frac{d^2 r}{dt^2} = -\frac{\left\{\frac{du}{d\theta}\right\}}{d\theta}\cdot\frac{d\theta}{dt}\cdot\frac{L}{m} = -\frac{d^2u}{d\theta^2}\cdot\frac{L}{m}\cdot\dot{\theta} = -\frac{d^2u}{d\theta^2}\cdot\frac{L}{m}\cdot\frac{L}{m}u^2$$

$$= -\frac{d^2u}{d\theta^2}\frac{L^2}{m^2}u^2$$

en (c)

$$-\frac{d^2u}{d\theta^2}\cdot\frac{L^2}{m^2}\cdot u^2 = -GMu^2 + \frac{L^2}{m^2}u^3 \Big/ \cdot -\frac{m^2}{L^2u^2}$$

$$\frac{d^2u}{d\theta^2} = \frac{GMm^2}{L^2} - u$$

$$\frac{d^2u}{d\theta^2} + u = \frac{GMm^2}{L^2} \Rightarrow \boxed{u(\theta) = \frac{GM}{L^2}m^2 + A\cos(\theta)}$$

donde $r = \frac{1}{u}$