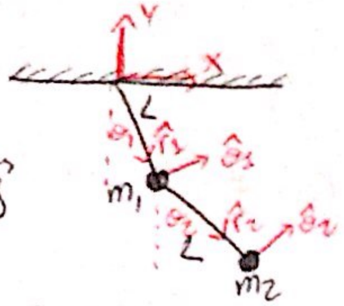
Pauta Examen 2014

Prof: Patricio Cordero
Aux: Sergio Cofré M.

P3



$\downarrow \vec{g}$

$m_1 = m_2 = m$

a) Las restricciones son geométricas, dadas por los largos de las cuerdas. Así sea (x_1, y_1) la posición de la primera masa y (x_2, y_2) la segunda:

$$\left.\begin{aligned} x_1^2 + y_1^2 &= L^2 \\ (x_1 - x_2)^2 + (y_1 - y_2)^2 &= L^2 \end{aligned}\right\} \text{Restricciones del problema.}$$

b) Escogiendo las coordenadas descritas en el dibujo vemos que:

$$\vec{r}_1 = L\hat{r}_1 \quad ; \quad \vec{V}_1 = L\dot{\theta}_1\hat{\theta}_1$$

$$\vec{r}_2 = L\hat{r}_1 + L\hat{r}_2 \quad ; \quad \vec{V}_2 = L\dot{\theta}_1\hat{\theta}_1 + L\dot{\theta}_2\hat{\theta}_2$$

Así $T = T_1 + T_2$, $T_1 = \frac{1}{2} m L^2 \dot{\theta}_1^2$

$$T_2 = \frac{1}{2} m L^2 (\dot{\theta}_1\hat{\theta}_1 + \dot{\theta}_2\hat{\theta}_2)\cdot(L\dot{\theta}_1\hat{\theta}_1 + L\dot{\theta}_2\hat{\theta}_2)$$

Notamos que $\hat{\theta}_1\cdot\hat{\theta}_1 = \hat{\theta}_2\cdot\hat{\theta}_2 = 1$ y $\hat{\theta}_1\cdot\hat{\theta}_2 = \hat{\theta}_2\cdot\hat{\theta}_1 = \cos(\theta_2 - \theta_1)$

$$\Rightarrow T = \frac{1}{2} m L^2 \left(2\dot{\theta}_1^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2-\theta_1) + \dot{\theta}_2^2\right)$$

Del mismo modo

$$V = V_1 + V_2 = \overset{\text{ojo!}}{-}(\underbrace{mg\cos\theta_1}_{m_1} + \underbrace{mg\cos_1 + mg\cos\theta_2}_{m_2})L$$

$$\Rightarrow V = -mgL(2\cos\theta_1 + \cos\theta_2)$$

Luego $\mathcal{L} = T - V = \frac{1}{2} m L^2 (2\dot{\theta}_1^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2-\theta_1) + \dot{\theta}_2) + mgL(2\cos\theta_1 + \cos\theta_2)$

c) Notamos que tenemos 4 grados de libertad $(\theta_1, \dot\theta_1, \theta_2, \dot\theta_2)$

Así $q = \theta_1, \theta_2$ y $\dot q = \dot\theta_1, \dot\theta_2$

Ecuación Euler-Lagrange: $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot q}\right) - \frac{\partial \mathcal{L}}{\partial q} = 0$

$q = \theta_1$

$$\frac{d}{dt}\left(2mL^2\dot\theta_1 + mL^2\dot\theta_2\cos(\theta_2-\theta_1)\right) + mL^2\dot\theta_1\dot\theta_2\sin(\theta_2-\theta_1) + 2mgL\sin\theta_1 = 0$$

$q = \theta_2$

$$\frac{d}{dt}\left(mL^2\dot\theta_2 + mL^2\dot\theta_1\cos(\theta_2-\theta_1)\right) - mL^2\dot\theta_1\dot\theta_2\sin(\theta_2-\theta_1) + mgL\sin\theta_2 = 0$$

Para pequeñas oscilaciones podemos considerar $\sin\phi \approx \phi$ y $\cos\phi \approx 1$ con $\phi \ll 1$. NO! es correcto decir que $\dot\phi \approx 0$.

$\Rightarrow \cos(\theta_2-\theta_1) \approx 1$; $\sin(\theta_2-\theta_1) \approx \theta_2-\theta_1$; $\sin(\theta_1) \sim \theta_1$; $\sin(\theta_2) \sim \theta_2$

$$\Rightarrow \frac{d}{dt}\left(2mL^2\dot\theta_1 + mL^2\dot\theta_2\right) + mL^2\dot\theta_1\dot\theta_2(\theta_2-\theta_1) + 2mgL\theta_1 = 0$$

$$\Rightarrow 2mL^2\ddot\theta_1 + mL^2\ddot\theta_2 = -\left(2mgL\theta_1 + mL^2\dot\theta_1\dot\theta_2(\theta_2-\theta_1)\right)$$

$$\Rightarrow 2\ddot\theta_1 + \ddot\theta_2 = -\left(\frac{2g}{2}\theta_1 + \dot\theta_1\dot\theta_2(\theta_2-\theta_1)\right) \quad (1)$$

$$\Rightarrow \frac{d}{dt}\left(mL^2\dot\theta_2 + mL^2\dot\theta_1\right) - mL^2\dot\theta_1\dot\theta_2(\theta_2-\theta_1) + mLg\theta_2 = 0$$

$$\Rightarrow mL^2(\ddot\theta_2 + \ddot\theta_1) = -mLg\theta_2 + mL^2\dot\theta_1\dot\theta_2(\theta_2-\theta_1)$$

$$\Rightarrow \ddot\theta_2 + \ddot\theta_1 = -\frac{g}{2}\theta_2 + \dot\theta_1\dot\theta_2(\theta_2-\theta_1) \quad (2)$$

Así calculamos $(1)+(2)$

$$\Rightarrow 3\ddot\theta_1 + 2\ddot\theta_2 = -\left(\frac{2g}{2}\theta_1 + \frac{g}{2}\theta_2\right)$$

$$\Rightarrow \boxed{3\ddot\theta_1 + 2\ddot\theta_2 = -\frac{g}{2}(2\theta_1 + \theta_2)}$$

Perdón por los borrones !!