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A Mathematical Formulation of the Ricardian System*

Since his own time, David Ricardo has always occupied a privileged place among economists, even in periods when economic analysis has been developing along paths very different from the ones he pursued. It has never been easy, however, for Ricardo's many interpreters, to state his complete system in a rigorous and concise form, and the reason lies in the peculiarity of some of the concepts he used which are not always defined in an unambiguous way. These concepts have encountered strong criticisms almost at any time, while—on the other hand—the bold analyses they made possible were exerting a sort of fascinating attraction.

In this paper, criticism is left aside and the more constructive approach is taken of stating explicitly the assumptions needed in order to eliminate the ambiguities. Then, the Ricardian system is shown to be very neat and even suitable for a mathematical formulation, with all the well-known advantages of conciseness, rigour and clarity. The task is undertaken in sections 4 to 9, which form the main part of the paper (part II). To avoid digressions and lengthy references there, the difficulties Ricardo was faced with, and the basic features of his theories, are briefly reviewed in the first three introductory sections (part I).

I

1. Theory of Value

The theory of value represents the most toilsome part of Ricardo's theoretical system and in our mathematical formulation it will entail the crudest assumptions. At the time it was put forward, the theory soon became the main target of the criticisms, which Ricardo tried to answer by re-writing twice (in the second and in the third edition) the chapter ' on value ' of his *Principles*.[1] No fundamental change was introduced, however, and the three versions represent different ways of framing (in the light of the criticisms) a theory of value which remains essentially the same[2].

* I am grateful to Mr. Kaldor, Prof. Modigliani and Mr. Sraffa for comments and criticism, and to Dr. James Message for a most helpful suggestion in the third section of the appendix.

[1] David Ricardo, *On the Principles of Political Economy and Taxation* (cit. as *Principles*). All references to Ricardo's works in this paper refer to the edition prepared by Piero Sraffa, *The Works and Correspondence of David Ricardo*, in 10 volumes, Cambridge University Press, 1951, (cit. as *Works*).

[2] This is a view to which recently Mr. Sraffa has given full support (*Works*, vol. I, Introduction, pp. XXXVII and ff.). Fragments of an early version of the Ricardian theory of value can be traced in Ricardo's early writings and in some letters (see the evidence given by Mr. Sraffa, *Works*, p. XXXI). It seems that Ricardo tried at the beginning to measure the relevant variables of his system in terms of a main agricultural commodity, namely corn, claiming that this commodity has the property of being both the capital and the product and, therefore, makes it possible to determine the ratio of profit to capital in physical terms without any question of evaluation. This position was, however, very vulnerable and will not be considered in this paper, as Ricardo abandoned it long before writing the *Principles*.

The theory is fundamentally based on the cost of production measured in terms of quantity of labour. Utility[1] is considered to be absolutely essential to, but not a measure of, exchangeable value. To commodities which derive their value from " scarcity alone "[2] (e.g., rare paintings) only a few words are devoted—they are not considered relevant for economic analysis ; Ricardo is concerned only with commodities which are the outcome of a process of production. He begins by restating Adam Smith's proposition that " in the early stages of society, the exchangeable value of commodities . . . depends . . . on the comparative quantity of labour expended on each ".[3] Then, he takes a new and striking step by asserting that *the mentioned proposition is valid in general* and not only in the early stages of society, as Smith claimed. His argument may be roughly expressed in the following way. Suppose two commodities, *A* and *B*, the first of which requires the work of one worker for one year to be produced and the second the work of two workers for one year (the capital employed being just the amount of wages to be anticipated to the workers). Whatever the rate of profit may be, either 10% or 20% or 30%, its amount on the second commodity always is twice as much as on the first commodity ; hence the relative price of the two goods always comes out as equal to the ratio of the quantities of labour required to obtain each of them.[4] If a "commodity could be found, which now and at all times required precisely the same quantity of labour to produce it, that commodity would be of an unvarying value"[5]: it would be an *invariable standard* in terms of which the value of all commodities could be expressed.

This formulation of the theory, of course, did not remain unchallenged. Strong objections were immediately raised (by Malthus, McCulloch, Torrens and others) which may be summarised as follows. Let us suppose, returning to the mentioned example, that the production of commodity *B* requires the work of one worker for two years instead of the work of two workers for one year. In this case, Ricardo's principle no longer applies because, owing to the profits becoming themselves capital at the end of the first year, a change in the rate of profit *does* imply a change in the relative price of the two commodities, even though the relative quantities of labour required by them remain the same. Ricardo could not ignore these objections and already in the first edition of the *Principles* he allowed for some exceptions to his general rule. All exceptions—as he later explained in a letter—" come under one of time ",[6] but he preferred discussing them, in the third edition of the *Principles*, under three groups (i. different proportions of fixed and circulating capital, ii. unequal durability of fixed capital, iii. unequal rapidity with which the circulating capital returns to its employer). However, while allowing for exceptions, Ricardo kept the fundamentals of his theory and tried to overcome the objections by appealing to the order of magnitude of the deviations caused by the exceptions, which he considered as responsible only for minor departures from his general rule. In the previous example, for instance, the modification introduced by the possibility that the same quantity of labour on *B* might be employed in one year or in two different years amounts simply to the effects caused by the amount of profit to be calculated on the wages of the first year. Ricardo holds that this is a difference of minor importance.[7] Therefore, the conclusion is,

[1] Needless to say, the term " utility " has for Ricardo, and in general for the Classics, a different meaning than for us to-day. It simply refers to the " value in use " of a commodity as opposed to its " value in exchange ". See *Principles*, p. 11.

[2] *Principles*, p. 12.

[3] *Principles*, p. 12.

[4] *Principles*, pp. 24 and ff.

[5] *Principles*, version of editions 1 and 2, see p. 17, footnote 3.

[6] Letter to McCulloch, *Works*, vol. VIII, p. 193.

[7] *Principles*, pp. 36 and ff.

the theory of value as stated in terms of quantities of labour, and independently of the distribution of commodities among the classes of the society, does hold, if not exactly, at least as a very good approximation.[1] With this premise, Ricardo considers as " the principal problem of Political Economy " that of determining " the laws which regulate the distribution ".[2]

2. Theory of Distribution

The participants in the process of production are grouped by Ricardo in three classes: landlords who provide land, capitalists[3] who provide capital and workers who provide labour. Total production is entirely determined by technical conditions but its division among the three classes—under the form of rent, profit and wages—is determined by the inter-action of many technical, economic and demographic factors. All Ricardo's analysis on this subject refers to what he calls the *natural* prices of rent, profits and wages. Divergencies of market prices from their natural level are considered only as temporary and unimportant deviations.

Rent, namely " that portion of the produce of the earth which is paid to the landlords for the use of the original and indistructible power of the soil "[4] is determined by technical factors. The technical property that different pieces of land have different fertility and that successive applications of labour to the same quantity of land yield smaller and smaller amounts of product (law of diminishing marginal returns) makes of rent a *net gain* for the landlords. Therefore, rent does not enter Ricardo's theory of value—it is a deduction from the total product. The value of commodities is determined by the quantity of labour employed on the marginal portion of land—that portion of land which yields no rent.

Wages are not related to the contribution of labour to the process of production, as in the modern theories they normally are. Like all economists of his time, Ricardo relates the level of wages to the physiological necessity of workers and their families to live and reproduce themselves. He is convinced that in any particular state of society there exists a *real* wage-rate (so to speak, a certain *basket of goods*) which can be considered as the " natural price of labour ". It need not necessarily be at a strict *subsistence level*[5] (the minimum physiological necessities of life) ; but at that level which in a given country and in a given state of society, besides allowing workers to live, induces them to perpetuate themselves " without either increase or diminution ".[6] When capitalists

[1] With the acceptance of criticisms between the first and the third edition of the Principles, also the choice of a " standard of value " became more difficult. Ricardo reacted to the complication by changing his definitions. In the first edition of the *Principles* he regarded as " standard " a commodity which would require at any time the same amount of unassisted labour (unassisted by capital) ; in the third edition he mentions a " commodity produced with such proportions of the two kinds of capital (fixed and circulating) as approach nearest to the average quantity employed in the production of most commodities ". (*Principles*, p. 63 and p. 45; see also *Works*, Introduction by Mr. Sraffa, vol. I, p. XLII and ff.). Ricardo considered one year a good average and thought that perhaps gold could be the commodity that most closely approaches the requirement of an *invariable standard.* (*Principles*, p. 45.)

[2] *Principles*, p. 5.

[3] Ricardo calls them alternatively " farmers " or " manufacturers " according as he refers to agricultural or to industrial capitalists.

[4] *Principles*, p. 67.

[5] " The natural price of labour—Ricardo says—varies at different times, in the same country, and very materially differs in different countries . . . it essentially depends on the habits and customs of the people . . . Many of the conveniences—Ricardo adds—now enjoyed in an English cottage would have been thought luxuries in an earlier period of our history ". (*Principles*, pp. 96-97).

[6] *Principles*, p. 93.

accumulate capital, demand for labour increases and the market wage-rate rises above its natural level. However, Ricardo believes that such a situation cannot be other than a temporary one because, as the conditions of workers become " flourishing and happy ", they " rear a healthy and numerous family "[1] and the growth of population again brings back the real wage-rate to its *natural* level. It is very impressive to notice how strongly Ricardo is convinced of the operation of this mechanism. To be precise, he always speaks of a process which will operate " ultimately " but the emphasis on it is so strong that his analysis is always carried on *as if* the response were almost immediate.

Profits, finally, represent a residual. Rent being determined by the produce of the marginal land put into cultivation, and the wage rate by non-economic factors, what remains of the total production is retained, under the form of profit, by the capitalists, who are the organizers of the process of production. The capitalists are assumed to be always intent on moving their capital towards any sector of the economy that shows a tendency to yield a rate of profit above the average. This behaviour ensures the equalization of the rate of profit (after risk) all over the economy.

3. Theory of Economic Growth

Economic growth is brought about essentially by the capitalists. The three classes in which Ricardo divides the society have different peculiar characteristics. Landlords are considered as an " unproductive class "[2] of wealthy people who become richer and richer, and consume almost all their incomes in *luxury goods*. Workers also consume everything they get but in a different kind of goods—" necessaries "—in order to live. Capitalists, on the other hand, are the *entrepreneurs* of the system. They represent the " productive class "[2] of the society. Very thrifty, they consume a small amount of what they obtain and devote their profits to capital accumulation.

The process of transforming profits into capital, however, cannot go on indefinitely. Owing to the decreasing marginal returns of new capital (and labour) applied to the same quantity of land, or to less fertile lands, rent increases over time, in real and in money terms, the *money* wage-rate increases too[3], and consequently the profit rate continuously falls.[4] When the rate of profit has fallen to zero, capitalists are prevented from accumulating any more ; the growth process stops and the system reaches a *stationary state*. As a matter of fact—Ricardo adds—the stationary state will be reached *before* the extreme point where all profits have disappeared because, at a certain minimum rate of profit, the capitalists will lose any inducement to accumulate. The final outcome (the stationary state) is postponed in time by new inventions and discoveries which increase the productivity of labour, but it is Ricardo's opinion that it will eventually be attained.

II

4. The " Natural " Equilibrium in a Two-Commodities System

It has been mentioned that Ricardo distinguishes two groups of commodities produced in the economy : " necessaries "—or, we may call them wage-goods—and " luxuries ". The most simple Ricardian system we can conceive of is, therefore, one where each of the two groups is reduced to one commodity. Let us begin with this case and make the following assumptions :

[1] *Principles*, p. 94.
[2] *Principles*, p. 270.
[3] How this happens will appear very clearly in the mathematical treatment of the following sections.
[4] *Principles*, especially chapters VI and XXI.

(i) the system produces only one type of wage-good, let us call it corn ;

(ii) to produce corn, it takes exactly one year ;

(iii) capital consists entirely of the wage bill ; in other words, it is only circulating capital, which takes one year to be re-integrated ;

(iv) there does exist an invariable standard of value, namely a commodity, let us call it gold—a luxury-good—,which at any time and place always requires the same quantity of labour to be produced. Its process of production also takes one year. Prices are expressed in terms of such a commodity and the monetary unit is that quantity of gold which is produced by the labour of one worker in one year.

The Ricardian system can now be stated in terms of equations. Taking the quantity of land in existence as given and supposing that its technical characteristics (fertility and possibilities of intensive exploitation) are known, the production of corn can be expressed by a technical production function, which we may assume to be continuously differentiable :

(1) $X_1 = f(N_1)$ where : X_1 = physical quantity of corn produced in one year ;

N_1 = number of workers employed in the corn production ;

with the following properties:

(1*a*) $f(0) \geqslant 0$

(1*b*) $f'(0) > \bar{x}$ where : $\bar{x}$ = natural wage-rate in terms of corn ;

(1*c*) $f''(N_1) < 0$.

The first inequality means that when no labour is employed, land is supposed to produce either something or nothing at all (negative production is excluded). The meaning of (1*b*) is that, at least when the economic system begins to operate and workers are employed on the most fertile piece of land, they must produce more than what is strictly necessary for their support, otherwise the whole economic system would never come into existence. Finally, (1*c*) expresses the law of decreasing marginal returns.

The production function for gold is much simpler :

(2) $X_2 = \alpha N_2$ where : X_2 = physical quantity of gold produced in one year ;

N_2 = number of workers employed in the production of gold ;

α = physical quantity of gold produced by one worker in one year ($\alpha > 0$).

The following equations are self-explanatory :[1]

(3) $N = N_1 + N_2$ where : N = total number of workers ; N_1 = agricultural workers ; N_2 = workers in the gold industry ;

(4) $W = N x$ W = total wage-bill, in terms of physical units of corn ; x = real wage-rate (corn) ;

(5) $K = W$ K = physical stock of capital (corn) ;

(6) $R = f(N_1) - N_1 f'(N_1)$ R = yearly rent, in real terms (corn) ;

(7) $P_1 = X_1 - R - N_1 x$ P_1 = yearly total profits, in real terms (corn), in the corn producing sector.

All variables introduced so far are in physical terms. Turning now to the determination of values, we have

(8) $p_1 X_1 - p_1 R = N_1$ where : p_1 = price of corn ;

(9) $p_2 X_2 = N_2$ p_2 = price of gold.

Equations (8) and (9) are very important in the Ricardian system. They state that the value of the yearly product, *after deduction of rent*, is determined by the quantity of labour required to produce it. In our case, owing to the definition of the monetary unit, the value of the product, after paying rent, is exactly equal to the number of workers employed. From (1), (2) and (6), equations (8) and (9) may be also written

(8*a*) $$p_1 = \frac{N_1}{X_1 - R} = \frac{1}{f'(N_1)}$$

(9*a*) $$p_2 = \frac{1}{\alpha}.$$

Profits in the gold industry and total profits in the economy emerge as

(10) $p_2 P_2 = p_2 X_2 - N_2 p_1 x$ where : P_2 = profits, in terms of physical units of gold, in the gold industry ;

(11) $\pi = p_1 X_1 + p_2 X_2 - p_1 R - p_1 W$ π = total profits, in terms of the standard of value.

After substituting from (1)-(10), equation (11) may be also written

(11*a*) $\pi = (N_1 + N_2)(1 - x p_1)$.

[1] Equation (6) may not appear so evident as the other equations. Let me state, therefore, an alternative way of writing it. As explained in section 2, rent represents for Ricardo a *surplus*, a net gain for the owners of the more fertile lands with respect to the owners of the marginal land (the land which yields no rent). Therefore, when N_1 workers are employed on land, the resulting total rent can be expressed as a sum of all the *net gains* of the non-marginal land-owners. In analytical terms :

(6*a*) $$R = f(0) + \int_0^{N_1} [f'(y) - f'(N_1)]\, dy$$

where $f(0)$, from (1*a*), is the produce that the land-owners can get from land without renting it, i.e. without any labour being employed. By solving the integral appearing in (6*a*), we obtain

$$R = f(0) + f(N_1) - f(0) - N_1 f'(N_1)$$

which is exactly equation (6).

At this point, the equations contain a theory of value and a theory of distribution but not yet a theory of expenditure. Since Ricardo assumes that all incomes are spent (Say's law), to determine the composition of total expenditure only one equation is necessary in the present model, specifying the production of one of the two commodities. Then the quantity produced of the other commodity turns out to be implicitly determined, as *total* production has already been functionally specified. The Ricardian theory is very primitive on this point. Workers are supposed to spend their income on necessities (corn, in our case) capitalists on capital accumulation (corn again, in our case) and land-owners on luxuries. Hence the determining equation is [1]

(12) $p_2 X_2 = p_1 R.$

Let us also write

(13) $w = p_1 x$ where : w = monetary wage-rate ;

(14) $r = \dfrac{\pi}{p_1 K}$ r = rate of profit.

So far 16 variables have appeared : $X_1, X_2, N_1, N_2, N, W, x, K, R, P_1, P_2, \pi, p_1, p_2, w, r,$ but only 14 equations. Two more equations are needed in order to determine the system. In a situation which Ricardo considers as *natural*, the following two data have to be added:

(15) $x = \bar{x} > 0$ where : $\bar{x}$ = *natural* real wage-rate, defined as that wage-rate which keeps population constant ;

(16) $K = \bar{K} > 0$ $\bar{K}$ = given stock of capital at the beginning of the year.

The system is now complete and determinate.[2] It can be easily demonstrated (see the appendix) that properties (1*a*), (1*b*), (1*c*) and the inequalities put on (15)-(16) are sufficient conditions to ensure the existence and uniqueness of non-negative solutions. We may consider, therefore, the system of equations (1)-(16) as defining the *natural* equilibrium of the Ricardian system.[3]

5. Some Characteristics of the Ricardian System

Already at this stage, the system of the previous section clearly shows some of the most peculiar characteristics of the Ricardian model. First of all, it contains a theory of value which is completely and (owing to our explicit assumptions) rigorously independent

[1] To be precise, we should allow for a minimum of necessities to be bought by the land-owners. This *minimum*, however, introduces only a constant into the analysis without modifying its essential features. For simplicity, therefore, the procedure is followed of neglecting the constant, which amounts to considering the minimum as negligible and supposing that the whole rent is spent on luxuries. Similarly, a minimum of luxuries might be allowed to be bought by the capitalists. This *minimum* also will be considered as negligible.

[2] It may be interesting to notice that equations (1), (4), (5), (6), (7), (15) and (16), taken by themselves, form an extremely simplified but determined Ricardian system expressed in terms of corn, where any question of evaluation has not yet arisen, corn being the single commodity produced. This is the system which has been used by Mr. Kaldor in his article " Alternative Theories of Distribution," *Review of Economic Studies*, 1955-56.

[3] To justify the terminology, let me mention that in his article " On the Notion of Equilibrium and Disequilibrium " (*Review of Economic Studies*, 1935-36), Professor Ragnar Frisch distinguishes two types of equilibria : *stationary* and *moving*. The *natural* equilibrium of the Ricardian system is not a stationary one, as will be seen in a moment ; it belongs to the *moving* type. Professor Frisch, in that article, describes a somewhat similar situation for the Wicksellian *normal* rate of interest.

of distribution. From equations (8*a*) and (9*a*), it appears that the value of commodities depends exclusively on technical factors (the quantity of labour required to produce them) and on nothing else.

Moreover, the system shows that wage-goods and luxury-goods play two different roles in the system. The production function for the wage commodity turns out to be of fundamental importance, while the conditions of production of the luxury-goods, expressed by α, have in the system a very limited influence. As can be easily found out (see also the appendix), the solutions for *all* variables, except p_2, depend on the function $f(N_1)$ or on its first derivative, while the constant α only enters the solutions for X_2 and p_2. As a consequence, the rate of profit and the money wage-rate are determined by the conditions of production of wage-goods and are entirely independent of the conditions of production of luxury goods.[1] It follows, for example—to mention one problem of concern to Ricardo—that a tax on wage-goods would affect all the participants in the process of production by changing both the money wage-rate and the rate of profit (as can be inferred from equations (13*a*) and (14*a*)), while a tax on luxury-goods would affect only the purchasers of these goods because it leaves the rates of profit and of wages entirely unaffected.[2]

6. The Market Solutions and the Attainment of the "Natural" Equilibrium

Ricardo admits that the market outcomes may not necessarily coincide with those of his " natural " equilibrium, but he considers two types of mechanisms which make the former converge towards the latter. First, he mentions the behaviour of the capitalists, whose readiness to move their capital towards the most profitable sectors of the economy always cause the rates of profit to equalize in all sectors. Secondly, he considers the increase of the working population in response to increases in wages. About the first of these two processes, Ricardo does not really say much more than what is said above. He does not find it useful to enter into complicated details (and in this case they would have been very complicated indeed for him, who did not possess a demand theory). Simply he allows for the process and carries on his analysis (the system (1)-(16)) on the assumption that the equalization of the rates of profit has already been permanently achieved. On the other hand, his analysis is more explicit, and can be clearly formulated, on the second type of mechanism.

At the beginning of our hypothetical year, what is really given (besides capital) is not the wage-rate but the number of workers. Therefore, the solutions determined by the market (supposing the rates of profit already equalized) are given by the system (1)-(14), (16) plus the following equality (replacing (15)):

[1] This can be seen more clearly by re-writing (13) and (14) after substitution from (4), (8*a*), (11*a*) (15). We obtain:

(13*a*) $$w = \frac{\bar{x}}{f'(N_1)}\,;$$ (14*a*) $$r = \frac{f'(N_1)}{\bar{x}} - 1.$$

[2] The independence of the rate of profit from the conditions of production of luxury-goods is a property of all the theoretical models which use the distinction between wage- and luxury-goods. In plain words, it is due to the peculiarity that wage-goods are necessary to produce any type of goods, while luxury-goods are not. Mr. Sraffa pointed out to me that the property was first discovered by Ladislaus von Bortkiewicz (*Zur Berichtigung der Grundlegenden theoretischen Konstruktion von Marx im dritten Band des 'Kapital'*, in " Jahrbücher für Nationalökonomie und Statistik," July 1907. An English translation can be found as an appendix to the volume *Karl Marx and the Close of his System*, by E. Böhm-Bawerk and *Böhm-Bawerk's Criticism of Marx*, by R. Hilferding, translated and edited by P. M. Sweezy, New York 1949). From our mathematical formulation, the property comes out very simply and clearly.

(15a) $N = \bar{N}$ where : $\bar{N}$ = number of workers at the beginning of the year.

The system is again complete and determinate but the wage-rate has now become a variable and has a solution (the market solution). Ricardo is firmly convinced that this solution can only be a temporary and unstable one because, if it comes out different from the *natural* wage-rate ($\bar{x}$), the population will adjust itself in such a way as to bring the two rates together. Analytically, the mechanism may be expressed as follows:

(17) $$\frac{dN}{dt} = F(x - \bar{x})$$ where : t denotes time and x the wage-rate resulting from the system (1)-(14), (15), (16),

with the properties[1]:

(17a) $$F(0) = 0$$ $$F' > 0$$

which mean that population is stable when $x = \bar{x}$, and it increases (or decreases) when $x > \bar{x}$ (or $x < \bar{x}$).

The differential equation (17) with the properties (17a) is of a type which has been extensively studied by Professor Samuelson in connection with what he calls the *correspondence principle* between comparative statics and dynamics.[2] In our case, it can be easily demonstrated[3] that the dynamic movement for $x(t)$ generated by (17) is convergent towards $\bar{x}$ (the natural wage-rate), provided that $\frac{dx}{dN} < 0$, a condition which the system fulfils.[3] Hence, for x, only the *natural* solution $x = \bar{x}$ is a stable solution.

7. The Equilibrium of the Stationary State

The *natural* equilibrium examined in the previous sections is still not a stable state of affairs. Two other types of change are in operation in a Ricardian system as time goes on : (i) the improvements which take place in the technical conditions of production—in our terms, the shifts in time of the production function $f(N_1)$—, and (ii) the accumulation of capital by the capitalists, who add each year a substantial part of their profits to capital. Here again, Ricardo does not consider the first type of change—technical progress—in a systematic way (a characteristic which only to-day can be found in models of economic growth). He only points out that improvements in the technical conditions postpone in time the effects of the changes of type (ii). Since he thinks that these changes (capital accumulation) are—in order of magnitude—the more relevant ones, he concentrates his analysis on them, with the qualification that the effects he shows might be delayed, though not modified, by technical progress.

In analytical terms, capital accumulation represents another dynamic mechanism, in operation on the system already described, of the following type :

[1] The function F and the similar function Φ of the following section are supposed to be continuously differentiable.

[2] P. A. Samuelson, " The Stability of Equilibrium : Comparative Statics and Dynamics," *Econometrica*, April 1941, also *Foundations of Economic Analysis*, Harvard University Press, 1948, especially Chapter IX.

[3] A proof is given in the appendix,

(18) $$\frac{dK}{dt} = \Phi\left(\frac{1}{p_1}\pi\right)$$

or, from (8*a*) and (11*a*),

(19) $$\frac{dK}{dt} = \Phi\Big(N[f'(N_1) - x] \Big)$$

with the properties :[1]

(19*a*) $$\Phi(0) = 0$$ $$\Phi' > 0.$$

The differential equation (19) is of the same type as (17) and has now to be considered jointly with it. From mere inspection of the two equations, it can be seen that the solutions of the system at which the two dynamic mechanisms cease to operate (the stationary solutions) emerge when $x = \bar{x}$ and $\pi = 0$. Therefore, for any given state of technical knowledge, represented by the technical function $f(N_1)$, the stationary equilibrium is given by equations (1)-(14) plus the following two:

(15) $$x = \bar{x} > 0$$

(16*a*) $$\pi = 0.$$

In order to ensure the existence of non-negative solutions for this system, a somewhat stronger condition than (1*b*) is required, namely

(20) $$f'(0) > \bar{x} > f'(\infty) \qquad \text{where : } f'(\infty) = \lim_{N_1 \to \infty} f'(N_1).$$

The meaning is that there must be a certain point, as population increases, at which the product of the last worker put to work descends below the *natural* wage-rate (a condition which is implicit in Ricardo's arguments). If this condition were not satisfied, the system would expand indefinitely and the stationary state would never be reached. When (20) is satisfied, it is shown in the appendix that two types of solutions exist—one of them corresponds to the equality $f'(N_1) = \bar{x}$ and the other to the equality $N = 0$. The solutions of the second type, however, so called *trivial* (they mean that there is no economic system at all) are uninteresting and moreover they are *unstable*. On the other hand, the solutions corresponding to $f'(N_1) = \bar{x}$ are unique and perfectly *stable*. Therefore, the system necessarily converge towards them. When the situation they represent is attained, all dynamic mechanisms come to a standstill. The wage-rate is at its natural level (no longer disturbed by capital accumulation) and the rate of profit has fallen to zero. The system has reached a stable equilibrium—the Ricardian equilibrium of the stationary state.

8. The Process of Economic Growth

It has been shown in the foregoing sections that the Ricardian system contains many dynamic processes, although some of them are not systematically analysed. The two dynamic processes which are explicitly taken into consideration are convergent and lead to a stationary and stable state. Ricardo, however, investigates the properties of his system at a very particular stage of the whole movement, which he considers the relevant one. Most of his analysis is carried on *as if* the demographic mechanism has already fully worked through, while the capital accumulation process has not yet been completed. In other

[1] If a minimum rate of profit (let us call it $\bar{r}$) is considered necessary in order to induce capital accumulation, equation (18) has to be modified as follows $\frac{dK}{dt} = \Phi\left(\frac{1}{p_1}\pi - \bar{r}K\right)$. However, the conclusions drawn in the text remain the same, with the single modification that the stationary and stable point of convergency of the system instead of being at $\pi = 0$, is at $\pi = \bar{r}p_1K$.

words, he concentrates on describing the changing characteristics of his system in terms of *natural* behaviour of the variables in a process of capital accumulation.

In mathematical notations, this task becomes very easy. It is enough to consider the system (1)-(16) (in which the *natural* wage-rate has been permanently achieved) and to take the derivatives of each variable with respect to capital, which represents the datum—in the *natural* equilibrium—whose variation in time brings about economic growth. A substantial part of the Ricardian analysis is simply expressed by the signs of these derivatives. Let us consider them:

$$\frac{dN}{dK} = \frac{1}{\bar{x}} > 0 \qquad (21)$$

$$\frac{dN_1}{dK} = \frac{1}{\bar{x}}\left\{1 - \frac{f(N_1)\,f''(N_1)}{[f'(N_1)]^2}\right\}^{-1} > 0 \qquad (22)$$

$$\frac{dN_2}{dK} = \frac{1}{\bar{x}}\left\{1 - \frac{[f'(N_1)]^2}{f(N_1)\,f''(N_1)}\right\}^{-1} > 0 \qquad (23)$$

$$\frac{dX_1}{dK} = f'(N_1)\cdot\frac{dN_1}{dK} > 0 \qquad (24)$$

$$\frac{dX_2}{dK} = \alpha\,\frac{dN_2}{dK} > 0 \qquad (25)$$

$$\frac{dW}{dK} = 1 > 0 \qquad (26)$$

$$\frac{dR}{dK} = -N_1\cdot f''(N_1)\cdot\frac{dN_1}{dK} > 0 \qquad (27)$$

$$\frac{dp_1}{dK} = \frac{-f''(N_1)}{[f'(N_1)]^2}\cdot\frac{dN_1}{dK} > 0 \qquad (28)$$

$$\frac{dp_2}{dK} = 0 \qquad (29)$$

$$\frac{dw}{dK} = \bar{x}\cdot\frac{dp_1}{dK} > 0 \qquad (30)$$

$$\frac{dr}{dK} = \frac{f''(N_1)}{\bar{x}}\cdot\frac{dN_1}{dK} < 0. \qquad (31)$$

The derivatives have been obtained from the system (1)-(16) and the inequality signs follow from (1*b*), (1*c*), (15), (16), and from other previous inequalities among (21)-(31) themselves. Their economic meaning may be stated as follows. The number of workers (employment), all physical productions, the wage bill, total rent, the price of corn and the natural *money* wage-rate : all increase as long as the process of capital accumulation is going on. As an effect of the same process, the rate of profit constantly decreases. For Ricardo, it took, of course, a much longer process to show what here is demonstrated merely by the sign of a derivative.

Another variable whose response to capital accumulation particularly requires a long analysis in the *Principles*[1] is total profit. For this variable too, let us consider its derivative with respect to capital. From (11*a*) we obtain

[1] *Principles*, chapter VI.

(32) $$\frac{d\pi}{dK} = \frac{1}{f'(N_1)}\left[\frac{f'(N_1)}{\bar{x}} - 1 + K \cdot \frac{f''(N_1)}{f'(N_1)} \cdot \frac{dN_1}{dK}\right].$$

Now, from (1*c*) and (22), $f''(N_1) < 0$ and $\frac{dN_1}{dK} > 0$. Moreover $f'(N_1) > \bar{x}$ as long as the stationary state has not yet been attained. (At the stationary state $f'(N_1) = \bar{x}$). Therefore, the sign of (32), unlike all the others, is not independent of the amount of K. At the beginning of the process of capital accumulation, where $K = 0$, the third term into brackets vanishes and therefore $\frac{d\pi}{dK} > 0$. At the stationary state, where $f'(N_1) = \bar{x}$, the first two terms into brackets cancel out, and the third is negative, so that $\frac{d\pi}{dK} < 0$. In between, there must be at least one point of maximum total profits $\left(\text{where} \frac{f'(N_1) - \bar{x}}{\bar{x}} = -K \cdot \frac{f''(N_1)}{f'(N_1)} \cdot \frac{dN_1}{dK}\right)$ at which (32) changes its sign from positive to negative as capital accumulates.[1] Hence

$$\frac{d\pi}{dK} \gtreqless 0 \quad \text{according as to whether} \frac{f'(N_1)}{\bar{x}} - 1 \gtreqless -K \cdot \frac{f''(N_1)}{f'(N_1)} \cdot \frac{dN_1}{dK}$$

which may also be written $\frac{f'(N_1)}{\bar{x}} - 1 \gtreqless - \frac{E[f'(N_1)]}{EK}$, where the first member of the inequality represents the rate of profit (see equation (14*a*)) and the second member represents the *elasticity* of the marginal product from land with respect to capital.

Analytically, the possibility cannot be excluded of more than one point of maximum, in the sense that π might alternatively increase and decrease many times as capital accumulates. For such a possibility to realize, however, the third derivative of $f(N_1)$ must behave in a very peculiar way. Ricardo, of course, did not consider these complications; he explained the process by a long numerical and obviously non-rigorous example which allowed him to consider the normal case in which, as capital accumulation goes on, total profits increase up to a certain point and then decrease.[2] [3]

[1] The reader may easily verify that, as capital accumulates, profits in sector 1 and in sector 2 (namely the variables P_1 and P_2) behave exactly in the same way as total profit (π).

[2] *Principles*, pp. 110 and ff.

[3] While correcting the proofs, I was pointed out a recent paper by H. Barkai where a very simplified one-commodity Ricardian model is worked out in mathematical terms in order to analyse the movements of *relative shares* as capital accumulates (H. Barkai, " Ricardo on Factor Prices and Income Distribution in a Growing Economy," *Economica*, August 1959). Dr. Barkai uses a procedure which has some similarities with the one I have adopted in this section. His analysis, however, seems to me rather inaccurate. Without going into details here (among other things, his conclusions about the behaviour of the share of profits are not altogether correct) I shall only mention that Dr. Barkai's main contention is that the relative share of total wages in total product increases as an effect of capital accumulation (which is obvious, as the real wage-rate is constant and the production function is at diminishing returns) and that this result contradicts what Ricardo said on page 112 of his *Principles*, namely that as capital accumulates, " the labourer's . . . real share will be diminished " (Barkai's quotation). But where is the contradiction? Dr. Barkai's proof refers to the *relative share of total wages* in total product, while Ricardo is talking about the single *labourer's real share*, which is evidently a different thing.

This is a case where the interpretation of Ricardo is quite straightforward. However, as it is always so easy to be misled by particular passages in Ricardo's writings, let me recall the advice of Alfred Marshall: " If we seek to understand him [Ricardo] rightly, we must interpret him generously, more generously than he himself interpreted Adam Smith. When his words are ambiguous, we must give them that interpretation which other passages in his writings indicate that he would have wished us to give them. If we do this with the desire to ascertain what he really meant, his doctrines, though far from complete, are free from many of the errors that are commonly attributed to them ". (Alfred Marshall, *Principles of Economics*, Macmillan London, 8th ed. reset, page 670.)

9. Multi-Commodity Production

We are now in a position to drop the two-commodity assumption and extend our system of equations to the general case of multi-commodity production.As far as the wage-goods are concerned, the extension does not present particular difficulties, although it does emphasize the crudeness of Ricardo's assumptions. The economic theory of demand had not yet been developed, at that time, and there is no question of substitution among wage-goods in the Ricardian model. The *natural* wage-rate is represented by a fixed *basket of goods*, to be accepted as given by factors lying outside economic investigation. With this specification, the introduction in our system of any wage-good *i*, besides corn, introduces 8 more variables: X_i, N_i, W_i, R_i, P_i, K_i, p_i, x_i, but also 8 more equations of the types (1), (4), (5), (6), (7), (8), (15), (16). The system is again determinate and maintains its basic features already analysed in the previous sections. As a matter of fact, when the *natural* wage-rate is accepted as a fixed basket of goods, there is no gain at all, from an analytical point of view, in extending the system to include any number of wage-goods more than one. The whole structural character of the model is already given by the system of equations (1)-(16), provided that our interpretation of the single wage-commodity is modified in the sense of considering it as a composite commodity, made up of a fixed mixture of wage-goods.[1] The dynamic characteristics of the model also remain unchanged as they depend exclusively on the wage-goods part of the economy.

The problem becomes much more complicated when the extension to multi-commodity production is made for luxury-goods. Here, the introduction of each commodity l_j, besides the one which is used as a standard, introduces 4 more variables : X_{lj}, N_{lj}, p_{lj} P_{lj}, but only 3 more equations of the types (2), (9) and (10). Moreover, it changes equation (12) into the following one:

$$(12a) \qquad p_{l1}\, X_{l1} + p_{l2}\, X_{l2} + \ . \ . \ . \ p_{lj}\, X_{lj} + \ . \ . \ . \ p_{ln}\, X_{ln} = p_w\, R$$

where the subscript w stands for the composite wage-commodity and the subscripts l_j's stand for the luxury-goods.

Hence, for each luxury-commodity introduced besides the first, one more relation is needed in order to keep the system determinate. Ricardo *does not* provide this relation. Again the difficulty is that he does not have a theory of demand. The assumption of a natural wage-rate solves the problem for the workers (and by consequence for the capitalists) but leaves it still open for the landlords, whose possibilities of substituting one luxury-good for another and whose changes of tastes cannot be ruled out; and Ricardo does not rule them out. Have we to conclude, therefore, that the Ricardian system is indeterminate with respect to the luxury goods? It certainly is, but—interestingly enough—only for the particular variables X_{lj}'s. N_{lj}'s, P_{lj}'s, which are not really of much interest to an economist like Ricardo, *once their totals are determined.*

[1] Professor Samuelson, in his recent " Modern Treatment of the Ricardian Economy, (*The Quarterly Journal of Economics*, February and May, 1959), has been unable to grasp these properties of the Ricardian model. The reason seems to me that he has treated a Ricardian economy with a production function of the neo-classical type (see especially his appendix), which is inappropriate and is responsible for the conclusions he then criticises. Professor Samuelson argues that the classification of lands in order of fertility—namely, in our terms, the technical function $f(N_1)$ — is not an unambiguously determined one because, according to the type of produce which is considered, the classification, i.e. the function $f(N_1)$, may be different. This argument is valid in a neo-classical theoretical framework, where substitution among goods (in consumption and in production) is the main feature of the theory, but is irrelevant in a Ricardian type of analysis, which excludes substitution. When the proportion of the different produces is fixed, the classification of lands in order of fertility is a perfectly determined one.

To see this surprising property of the Ricardian system, let us suppose that n luxury-goods are produced. Then $4(n-1)$ new variables of the types X_{lj}, N_{lj}, P_{lj}, p_{lj}, and $3(n-1)$ new equations of the types (2), (9) and (10) are introduced in the already analysed system. Provisionally, let us write down n demand equations for the luxury goods :

$$
\begin{aligned}
X_{l1} &= \varphi_1 (p_w, p_{l1}, p_{l2} \ldots\ldots p_{ln}, R) \\
X_{l2} &= \varphi_2 (p_w, p_{l1}, p_{l2}, \ldots\ldots p_{ln}, R) \\
&\ \ \vdots \\
X_{ln} &= \varphi_n (p_w, p_{l1}, p_{l2}, \ldots\ldots p_{ln}, R).
\end{aligned} \tag{33}
$$

Equation (12a), which represents Say's law (namely, landlords spend all their income—no more and no less—on luxury goods), puts a restriction on the (33), so that one of the equations may be dropped. We are left with $(n-1)$ equations, which is the number necessary to determine the system. It is very interesting to notice now that the solutions for all the variables of the system, except the X_{lj}'s, P_{lj}'s, N_{lj}'s, are independent of the (33). In other words, the (33) are required only to determine the single physical productions, employments and profits in each particular luxury-goods sector but not to determine all other variables. *Whatever the demand equations for luxury-goods may be*, i.e., independently of them, all the variables referring to the wage-goods part of the economy, all prices, the rate of profit, and all the macro-economic variables of the system—like total employment, national income, total profits, total rent, total wages, total capital—are already determined by the system.

This is perhaps the most interesting outcome of the whole mathematical formulation attempted in this paper and it will be useful to remind the reader of the assumptions under which it has been reached : (i) perfect mobility of capital ; (ii) Say's law ; (iii) the assumption of circulating capital only, and of a one-year period for *all* processes of production. The last assumption, so stated, is too restrictive. As a matter of fact, it may be dropped and fixed capital introduced into the analysis without affecting the already attained conclusions, provided that the somewhat more general restriction is kept of supposing that all the sectors of the economy use fixed and circulating capital of the same durability and in the same proportions. *This is indeed the crucial assumption* : the determinateness of the whole Ricardian system itself depends on it, in an essential way.

Ricardo himself became aware of this limitation of his theoretical model in connection with the problem of the determination of total employment in the economy. He was disturbed by the discovery and, as a result, in the third edition of the *Principles*, he added the well-known chapter " on machinery ". The problem is that, when the mentioned crucial assumption holds, total employment in the economy, for any given amount of capital, is determined independently of the (33). But when the conditions of the assumption are not realized, total employment comes out different according to the way in which demand (and therefore capital) is distributed among the luxury goods sectors. Having realized this, Ricardo declared explicitly, in the added chapter, that he was mistaken earlier when he extended to the introduction of machinery (i.e., to the case where the proportions of fixed and circulating capital change) his general conclusions about total employment depending on total capital alone and not on how and where this capital is employed. This proposition, in the light of our formulation, appears quite obvious, but it has not appeared so to many of Ricardo's interpreters. Indeed, because of the assertions it contains, which seem to be in contradiction with the general conclusions following from the whole previous analysis, the chapter " on machinery " has always puzzled Ricardo's readers. The mathematical formulation of the present paper helps to clarify the issue. It shows that the entire Ricardian model stands on the assumption of a uniform composition of capital all over the economy. The problem of introduction of machinery exactly hypothesizes a violation of this assumption. Therefore, the general conclusions cannot be extended to this case. Looked at in these terms, the chapter " on machinery " appears, rather than a contradiction, an honest acknowledgement by Ricardo of the limitations of his theory.

10. Concluding Remarks

A few remarks may be drawn as a way of conclusion.

Ricardo's model is built on very crude assumptions. The most crucial of them is that all sectors of the economy use—we might say in more modern terms—the same period of production. This was just the point against which his contemporary critics (especially Malthus) threw their most violent attacks. In their function as critics, they were right. The limits entailed by the assumption are relevant not only for the Ricardian theory of value—as has always been thought—but also for the determinateness itself of the whole system, as soon as the simple case of two-commodity production is departed from.

On the other hand, once the assumptions underlying the whole analysis have been explicitly defined, the system appears to be logically consistent and determinate in all its macro-economic features and even in its sectoral details, except for some particular sectoral variables in which Ricardo was not interested. A mathematical formulation of the model is possible, clarifies many issues—among others, those connected with the controversial chapter " on machinery "—and permits a representation of the Ricardian dynamic processes—in particular the process of economic growth—in a few rigorous and concise notations. The solutions of the *natural* system Ricardo was dealing with are shown to exist and to be unique but not stable. They reach a perfect stability only in the equilibrium of the stationary state.

The whole model, in its crudeness and simplicity, appears remarkably complete and synthetic. Ricardo is always looking for fundamentals. Detailed relations are dealt with only in the light of basic tendencies—when they become too complicated and lead to difficulties, those relations which are thought to be less important are *frozen* by crude assumptions. Whether this is a fruitful methodological line to pursue is open to controversy. Later, neo-classical economists preferred a radically different line of approach. They abandoned too ambitious dynamic outlooks and instead started to analyse, in a complete way and in all its functional interrelationships, at least a more simplified (static) version of economic reality. The step which was supposed to follow, however,—that of passing to a dynamic analysis—has not come out as easy and spontaneous as was expected, and, in recent years, it has not been infrequent for economists, faced with urgent problems of economic development, to have second thoughts on the subject. In this light, the Ricardian analysis, with all the naiveté and the limits of its particular theories, appears less primitive now-a-days than it appeared some decades ago.

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APPENDIX

Existence and Uniqueness of Stable Solutions

It has been a widespread concern among mathematical economists in the last few decades not to be satisfied any longer (as economists used to be) with mere counting the number of equations and unknowns of their theoretical systems and to enquire more rigorously into the conditions for the existence, uniqueness and stability of the solutions. The task has not proved to be an easy one, as it normally entails mathematical notions and manipulations of a fairly highly sophisticated nature. In our case, fortunately, the proofs can be given in a relatively elementary way, except perhaps for the stability conditions.

1. *Existence and uniqueness of non-negative, non-trivial solutions.* The Ricardian system contains one single *functional* relation, the $f(N_1)$. Therefore, the fundamental step to solving it is to find the value of N_1 which satisfies the restrictions put by the system on the $f(N_1)$. In the system (1)-(15), (16*a*), we may start by taking (11*a*), substitute it into (16*a*) and obtain

$$N[f'(N_1) - \bar{x}] = 0.$$

This equation is satisfied either by $f'(N_1) = \bar{x}$ or by $N = 0$. The latter solution means that there is no economic system at all. Any theoretical representation of an economic system has this solution, but it is an uninteresting one—it represents the so-called *trivial* case. Evidently, the *relevant* solution is the other one. Let us prove therefore:

(i) that N_1^*—defined as the solution of the equation $f'(N_1) = \bar{x}$ — exists and is non-negative ;

(ii) that N_1^* is unique ; and finally,

(iii) that $f(N_1^*) \geqslant N_1^* f'(N_1^*)$. (The reason for this proof will appear in a moment.)

Proof (i). From (20), $f'(0)$ exists and is greater than $\bar{x}$; $f'(\infty)$ also exists and is smaller than $\bar{x}$. Since $\bar{x}$ is a positive constant, there must be a value $0 < N_1^* < \infty$, at which $f'(N_1^*) = \bar{x}$. Hence N_1^* exists and is non-negative.

Proof (ii). From (1*c*), $f''(N_1) < 0$, namely $f'(N_1)$ is a monotonic function. Since $\bar{x}$ is a constant, then (by a straightforward application of Rolle's theorem) N_1^* is unique.

Proof (iii). Call $G = f(N_1) - N_1 f'(N_1)$. Then $\frac{dG}{dN_1} = -N_1 f''(N_1) > 0$, namely G is a monotonically increasing function. Since $f(0) \geqslant 0$ and $N_1 \geqslant 0$, from (1*a*) and (i), then G is never negative, or $f(N_1) \geqslant N_1 f'(N_1)$ and, in particular, $f(N_1^*) \geqslant N_1^* f'(N_1^*)$.

By substituting now N_1^* into the system of equations (1)-(15), (16*a*), the solutions come out as:

(A1) $X_1 = f(N_1^*)$

(A2) $X_2 = \alpha \left[\frac{f(N_1^*)}{f'(N_1^*)} - N_1^*\right]$

(A3) $N = \frac{f(N_1^*)}{f'(N_1^*)}$

(A4) $N_1 = N_1^*$

(A5) $N_2 = \frac{f(N_1^*)}{f'(N_1^*)} - N_1^*$

(A6) $W = \bar{x}\frac{f(N_1^*)}{f'(N_1^*)}$

(A7) $K = \bar{x}\frac{f(N_1^*)}{f'(N_1^*)}$

(A8) $R = f(N_1^*) - N_1^* f'(N_1^*)$

(A9) $p_1 = \frac{1}{f'(N_1^*)}$

(A10) $p_2 = \frac{1}{\alpha}$

(A11) $w = \bar{x}\frac{1}{f'(N_1^*)}$

(A12) $P_1 = 0$

(A13) $P_2 = 0$

(A14) $r = 0$

(A15) $x = \bar{x}$

(A16) $\pi = 0.$

It follows that, if N_1^* exists is unique and non-negative and, moreover, if $f(N_1^*) \geqslant N_1^* f'(N_1^*)$, then all (A1)-(A16), namely the non-trivial solutions of the system, exist, are unique and are non-negative. The proofs have been given so far with reference to the equations (1)-(15), (16*a*). *A fortiori*, the solutions of any other system of equations (1)-(16), defined by a given $\bar{K}$ between 0 and K^*, exist, are unique and non-negative. For the system (1)-(16) the trivial solutions are even excluded by hypothesis as $\bar{K} > 0$. (The stars * are taken to denote the non-trivial solutions of the stationary equilibrium).

2. *The stability of the stationary equilibrium.* The stationary equilibrium is defined by the solutions of the system of equations (1)-(15), (16*a*). In order to find out whether it is stable, an investigation has to be made into the dynamic behaviour of the system when *displaced* from the equilibrium solutions. That behaviour is represented by the two differential equations (17) and (18). For a rigorous proof of stability, the two equations have to be considered jointly. Such a proof is given below but, as it entails a rather sophisticated mathematical treatment, it may be useful to give first a more simple proof which, although less rigorous, is intuitively easier to grasp and perhaps also more pertinent to the Ricardian logic.

The function (17) depends on the deviation of x from $\bar{x}$ and the function (18) on the deviation of $f'(N_1)$ from x. The two dynamic mechanisms are, so to speak, one on the top of the other. We may begin, therefore, by proving first, for a given $\bar{K}$, the convergency of the first dynamic process towards $\bar{x}$ and then substitute this stable solution into the second process and carry on a similar investigation on it, for a given $\bar{x}$.

Let us take equation (17) and expand it in a Taylor series around a value of N defined as $N+ = \frac{\bar{K}}{\bar{x}}$:

$$\frac{d(N - N+)}{dt} = F(0) + (N - N+) F'(0) \frac{dx}{dN} + \frac{(N - N+)^2}{2}\left[F''(0)\left(\frac{dx}{dN}\right)^2 + F'(0)\frac{d^2x}{dN^2}\right] +$$

Neglecting the terms of higher order than the first and recalling that $F(0) = 0$, the equation becomes

$$\frac{d\,(N - N+)}{dt} = (N - N+) \cdot F'(0) \cdot \frac{dx}{dN}.$$

This is a simple differential equation and its solution is[1]

$$N(t) = N+ + [N(0) - N+] \cdot \exp\left[F'(0) \cdot \frac{dx}{dN} \cdot t\right]$$ where $N(0)$ is the value of N at time zero.

Since $F'(0) > 0$ from (17*a*), a necessary condition for $N(t)$ to converge towards $N+$ (and therefore for $x(t)$ to converge towards $\bar{x}$) is $\frac{dx}{dN} < 0$. Now, from the system (1)-(14), (15*a*), (16), we have $\frac{dx}{dN} = -\frac{K}{N^2} < 0$. *The condition is fulfilled.* Hence, *the solution $x = \bar{x}$ is stable.*

By substituting now $x = \bar{x}$ into (19) and developing the same type of analysis, the necessary condition for K to converge towards K^*—defined as the stationary equilibrium solution for K—is

$$\frac{d}{dK}\left(\frac{1}{p_1}\pi\right) < 0.$$

Now, from (11*a*) we can write

$$\frac{d}{dK}\left(\frac{1}{p_1}\pi\right) = \frac{f'(N_1)}{\bar{x}} - 1 + N \cdot f''(N_1) \cdot \frac{dN_1}{dK}.$$

Since $f''(N_1)$ is negative and $f'(N_1)$ is greater or equal to $\bar{x}$ according as to whether $N_1 < N_1^*$ or $N_1 = N_1^*$, then condition $\frac{d}{dK}\left(\frac{1}{p_2}\pi\right) < 0$ is not satisfied when $N = 0$, while it is satisfied when $f'(N_1) = \bar{x}$. Hence, *the solutions of the system corresponding to $f'(N_1) = \bar{x}$ are stable, while the trivial solutions are unstable*—the system necessarily converges towards the first ones. As a conclusion, the system (1)-(15), (16*a*) has *stable solutions*. Such stable solutions are also *unique*.

3. *A more rigorous proof of stability.* Consider equations (17) and (18), representing the variations in time of N and of K. Since N_1 is a monotonically increasing function of N, the equations may be equally expressed in terms of N_1 (namely in terms of the wage-goods sector):

(A17) $$\frac{dN_1}{dt} = g(x - \bar{x})\,; \quad g(0) = 0\,; \quad g' > 0;$$

(A18) $$\frac{dK_1}{dt} = \varphi(P_1), \quad \text{where } K_1 = N_1 x\,; \quad \varphi(0) = 0\,; \quad \varphi' > 0.$$

[1] See any elementary treatise on differential equations or also R. G. D. Allen, *Mathematical Economics*, London 1956, chapter 5.

Our purpose is now to investigate the dynamic behaviour of the system in the vicinity of the stationary solutions $x = \bar{x}$ and $f'(N_1) = x$. Let us expand (A17) in a Taylor series around the value $\bar{x}$. Neglecting the terms of higher order than the first the equation may be written

$$\text{(A17}a\text{)} \qquad \frac{d(N_1 - N_1^*)}{dt} = (x - \bar{x})\, g'(0).$$

Equation (A18) is more complex. Let us first write it in terms of the same vaiiables entering (A17),

$$\frac{dK_1}{dt} = \frac{d(N_1 x)}{dt} = \varphi\left(N_1\,[f'(N_1) - x]\right).$$

By expanding also this equation in a Taylor series and neglecting the terms of higher order than the first we obtain

$$\text{(A19)} \qquad N_1 \cdot \frac{dx}{dt} + x \cdot \frac{dN_1}{dt} = \varphi'(0) \cdot \left\{N_1[f'(N_1) - x]\right\}.$$

Let us now express the variables in terms of deviations from their stationary solutions and utilize Taylor's theorem for the $f'(N_1)$. We have

$$[N_1^* + (N_1 - N_1^*)]\,\frac{d(x - \bar{x})}{dt} + [\bar{x} + (x - \bar{x})]\,\frac{d(N_1 - N_1^*)}{dt} =$$
$$\varphi'(0) \cdot \left\{[N_1^* + (N_1 - N_1^*)] \cdot [(N_1 - N_1^*) \cdot f''(N_1^*) - x + f'(N_1^*)]\right\}.$$

The squares of $(N_1 - N_1^*)$ and of $(x - \bar{x})$, and their products, represent magnitudes of second order and we may neglect them, re-writing the whole expression as

$$\text{(A20)} \quad N_1^* \frac{d(x - \bar{x})}{dt} + \bar{x}\,\frac{d(N_1 - N_1^*)}{dt} = \varphi'(0) \cdot N_1^* \cdot [(N_1 - N_1^*) \cdot f''(N_1^*) - x + \bar{x}] +$$
$$0\left\{(x - \bar{x})^2;\ (N_1 - N_1^*)^2;\ (x - \bar{x})\,(N_1 - N_1^*)\right\}$$

where the last term denotes the order of magnitude of the neglected products. Multiplying now (A17a) by $\bar{x}$ and subtracting it from (A20) we can at last write down our equations in a suitable form for an immediate solution

$$\frac{d(N_1 - N_1^*)}{dt} = (x - \bar{x}) \cdot g'(0) + 0\left\{(x - \bar{x})^2\right\}$$

$$\frac{d(x - \bar{x})}{dt} = (N_1 - N_1^*) \cdot \varphi'(0) \cdot f''(N_1^*) - (x - \bar{x}) \cdot \left[\frac{\bar{x}}{N_1^*}\, g'(0) + \varphi'(0)\right] +$$
$$0\left\{(x - \bar{x})^2;\ (N_1 - N_1^*)^2;\ (x - \bar{x})\,(N_1 - N_1^*)\right\}.$$

The solutions of this system of equations—apart from the neglected terms—take the form[1]

[1] See, for example, A. R. Forsyth, *A Treatise on Differential Equations*, London, 1921, pp. 342 and ff. In order to make the procedure easier to follow, I shall take here the same steps as Professor Samuelson in his already mentioned article in *Econometrica*.

$$N_1(t) = N_1^* + k_{11} e^{\lambda_1 t} + k_{12} e^{\lambda_2 t}$$
$$x(t) = \bar{x} + k_{21} e^{\lambda_1 t} + k_{22} e^{\lambda_2 t}$$

where the k's depend on the values of N_1 and x at time zero and the λ's are the roots of the characteristic equation

$$\begin{vmatrix} 0 - \lambda & g'(0) \\ \varphi'(0) \cdot f''(N_1^*) & -\frac{\bar{x}}{N_1^*} g'(0) - \varphi'(0) - \lambda \end{vmatrix} = 0$$

or

$$\text{(A21)} \qquad \lambda^2 + \lambda \left[\varphi'(0) + \frac{\bar{x}}{N_1^*} \cdot g'(0)\right] - g'(0) \cdot \varphi'(0) \cdot f''(N_1^*) = 0.$$

For the equilibrium to be stable the real part of λ must be necessarily negative, i.e.,

$$\text{(A22)} \qquad R(\lambda) < 0.$$

Now, since $\varphi'(0) > 0$, $g'(0) > 0$, and $f''(N_1) < 0$, (A21) can be written as

$$\text{(A23)} \qquad \lambda^2 + 2m\lambda + n^2 = 0 \qquad \text{where:}$$
$$m = \tfrac{1}{2}\left[\varphi'(0) + \frac{\bar{x}}{N_1} g'(0)\right];$$
$$n = \sqrt{-g'(0) \cdot \varphi'(0) \cdot f''(N_1^*)}.$$

Hence:

$$\lambda = -m \pm (m^2 - n^2)^{\frac{1}{2}}$$

from which it appears that the real part of λ is always negative, namely that condition (A22) is satisfied. Therefore, the *stationary equilibrium defined by the couple of solutions* $x = \bar{x}$ and $f'(N_1) = \bar{x}$ *is stable.*

A proof of the instability of the trivial solutions, characterized by $N_1 = 0$, can be given in an easier way because in this case the products involving N_1 itself—besides those involving $(x - \bar{x})$—are of second order of smallness and equation (A18) may be considered in isolation, as appears by re-writing (A19) as

$$N_1 \frac{d(x - \bar{x})}{dt} + (x - \bar{x} + \bar{x}) \frac{dN_1}{dt} = \varphi'(0)\, N_1[f'(0) - x + \bar{x} - \bar{x}].$$

and then, neglecting the squares of N_1 and of $(x - \bar{x})$ and their products,

$$\frac{1}{N_1} \frac{dN_1}{dt} = \frac{1}{\bar{x}} \varphi'(0)\, [f'(0) - \bar{x}] + 0 \left\{N_1^2;\ (x - \bar{x})^2;\ N(x - \bar{x})\right\}.$$

This is a simple differential equation whose solution—apart from the neglected terms—take the form

$$\text{(A24)} \qquad \log_n N_1 = k\,t + \log_n C$$

namely

$$\text{(A25)} \qquad N_1(t) = C\, e^{kt} \qquad \text{where: } k = \frac{1}{\bar{x}}\, \varphi'(0)\, [f'(0) - \bar{x}],$$

$$\text{and} \quad C = N_1(0).$$

Since $\frac{1}{\bar{x}}\, \varphi'(0)\, [f'(0) - \bar{x}] > 0$, then the solution (A25) is explosive, which means that *the stationary equilibrium defined by the solution* $N_1 = 0$ *is unstable.*