Ayudantia 2
Asymptotics Theory

Nicolas Tagle

Marzo 2013

Contents

1 Exercices **1**

1.1 Exercise 1: A special case for the relation between $\xrightarrow{d}$ and $\xrightarrow{p}$ 1

1.2 Exercise 2: Consistency of the variance estimator 2

1.3 Exercise 3: The Delta Method 2

1 Exercices

1.1 Exercise 1: A special case for the relation between $\xrightarrow{d}$ and $\xrightarrow{p}$

(Amemiya 3.3) Suppose $X_N \xrightarrow{d} \alpha$, where α is a constant. Prove that $X_N \xrightarrow{p} \alpha$

Solution

For any $\epsilon > 0$ we have that

$$\begin{aligned} P(|X_N - \alpha| \le \epsilon) \ge & \quad P(\alpha - \epsilon < X_N \le \alpha + \epsilon) \\ = & \quad P(X_N \le \alpha + \epsilon) - P(X_N \le \alpha - \epsilon) \end{aligned}$$

But $X_N \xrightarrow{d} \alpha$ implies that $P(X_N \le x) \longrightarrow 1(\alpha \le x)$ i.e. the cdf of a constant α. This implies that

$$P(X_N \le \alpha + \epsilon) \longrightarrow 1(\alpha \le \alpha + \epsilon) = 1$$

$$P(X_N \le \alpha - \epsilon) \longrightarrow 1(\alpha \le \alpha - \epsilon) = 0$$

hence,

$$P(|X_N - \alpha| \le \epsilon) \longrightarrow 1 \Longleftrightarrow P(|X_N - \alpha| > \epsilon) \longrightarrow 0$$

which are equivalent definitions of convergency in probability, hence $X_N \xrightarrow{p} \alpha$

1.2 Exercise 2: Consistency of the variance estimator

Show that if $\{x_n\}$ is an iid sequence of random variables with finite variance, then $\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$ is a consistent estimator of $Var(x_i)$.

Solution:

$$\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2$$

The first term converges n probability by the LLN to $E(x_i^2)$.

The average inside brackets converges in probability to $E(x_i)$.

By the Continuous Mapping Theorem, $\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2 \rightarrow_p E(x_i)^2$.

The result follows by CT.

1.3 Exercise 3: The Delta Method

Let the positive random vector $(X_n, Y_n)'$ be such that:
$\sqrt{n}\left(\left(\begin{array}{c} X_n \\ Y_n \end{array}\right) - \left(\begin{array}{c} \mu_x \\ \mu_y \end{array}\right)\right) \longrightarrow_d N\left(\left(\begin{array}{c} 0 \\ 0 \end{array}\right), \left(\begin{array}{cc} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{array}\right)\right)$. Find the joint asymptotic distribution of

$\left(\begin{array}{c} \ln X_n - \ln Y_n \\ \ln X_n + \ln Y_n \end{array}\right)$. What is the condition under which $\ln X_n - \ln Y_n$ and $\ln X_n + \ln Y_n$ are asymptotically independent?

Solution:
We are going to use the Delta Method for

$$\sqrt{n}\left(\left(\begin{array}{c} X_n \\ Y_n \end{array}\right) - \left(\begin{array}{c} \mu_x \\ \mu_y \end{array}\right)\right) \longrightarrow_d N\left(\left(\begin{array}{c} 0 \\ 0 \end{array}\right), \sum\right)$$

and

$$g\left(\begin{array}{c} X_n \\ Y_n \end{array}\right) = \left(\begin{array}{c} \ln X_n - \ln Y_n \\ \ln X_n + \ln Y_n \end{array}\right); \ \sum = \left(\begin{array}{cc} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{array}\right)$$

So we compute the gradient of g in order to get G_0.

$$\frac{\partial g}{\partial (X_n, Y_n)} \begin{pmatrix} X_n \\ Y_n \end{pmatrix} = \begin{pmatrix} \frac{1}{X_n} & -\frac{1}{Y_n} \\ \frac{1}{X_n} & \frac{1}{Y_n} \end{pmatrix}$$

so we can get G_0.easily

$$G_0 = \begin{pmatrix} \frac{1}{\mu_x} & -\frac{1}{\mu_y} \\ \frac{1}{\mu_x} & \frac{1}{\mu_y} \end{pmatrix}$$

so

$$\sqrt{n}\left(\begin{pmatrix} \ln X_n - \ln Y_n \\ \ln X_n + \ln Y_n \end{pmatrix} - \begin{pmatrix} \ln \mu_x - \ln \mu_y \\ \ln \mu_x + \ln \mu_y \end{pmatrix}\right) \longrightarrow_d N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, G_0 \sum G_0'\right)$$

where

$$G_0 \sum G_0' = \begin{pmatrix} \frac{\sigma_{xx}}{\mu_x^2} - \frac{2\sigma_{xy}}{\mu_x \mu_y} + \frac{\sigma_{yy}}{\mu_y^2} & \frac{\sigma_{xx}}{\mu_x^2} - \frac{\sigma_{yy}}{\mu_y^2} \\ \frac{\sigma_{xx}}{\mu_x^2} - \frac{\sigma_{yy}}{\mu_y^2} & \frac{\sigma_{xx}}{\mu_x^2} + \frac{2\sigma_{xy}}{\mu_x \mu_y} + \frac{\sigma_{yy}}{\mu_y^2} \end{pmatrix}.$$

Finally, the condition is $\frac{\sigma_{xx}}{\mu_x^2} = \frac{\sigma_{yy}}{\mu_y^2}$.