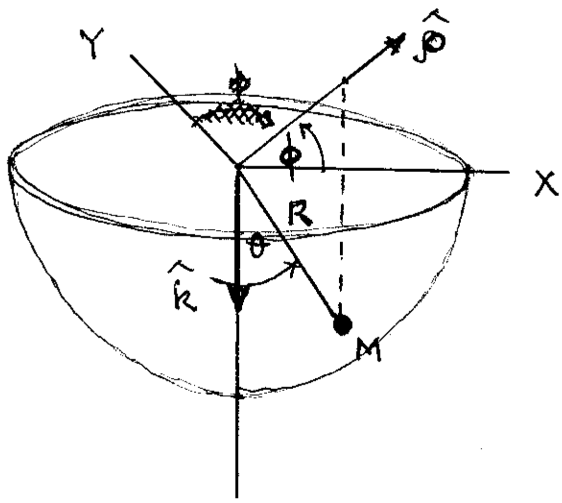
Solución :



En el sistema rotatorio, la ecu. de movimiento está dada por la ecu. maestra .

$$\vec{F} = M\vec{a}' + M\left(2\vec{\Omega}\times\vec{v}' + \vec{\Omega}\times(\vec{\Omega}\times\vec{r}')\right) .$$

Usamos el sistema esférico de la figura y nos guardamos la notación de primas, entendiendo que estamos en el sistema rotatorio .

$$\vec{r} = r\hat{r}(\theta,\phi) .$$

$$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\hat{r}}$$

pero $\dot{\hat{r}} = \vec{\omega}\times\hat{r}$

$$\vec{\omega} = \dot{\theta}\hat{\phi} + \dot{\phi}\hat{k} .$$

$$\therefore \boxed{\dot{\hat{r}} = \dot{\theta}\hat{\theta} + \dot{\phi}\sin\theta\,\hat{\phi} .}$$

$$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\dot{\phi}\sin\theta\,\hat{\phi} .$$

En nuestro caso:

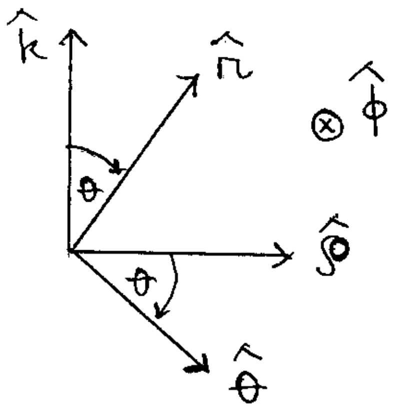
$$\dot{r} = 0 \longrightarrow r = R .$$

$$\therefore \quad \vec{r} = R\hat{r}$$

$$\dot{\vec{r}} = R\dot{\theta}\hat{\theta} + R\dot{\phi}\sin\theta\,\hat{\phi} .$$

Luego $\ddot{\vec{r}} = R\ddot{\theta}\hat{\theta} + R\dot{\theta}\dot{\hat{\theta}} + R(\ddot{\phi}\sin\theta + \dot{\phi}\dot{\theta}\cos\theta)\hat{\phi}$
$\qquad + R\dot{\phi}\sin\theta\,\dot{\hat{\phi}}$

pero: $\dot{\hat{\phi}} = \vec{\omega}\times\hat{\phi} = \dot{\phi}\hat{k}\times\hat{\phi} = -\dot{\phi}\hat{\rho} .$



Luego

$$\hat{\rho} = \sin\theta\,\hat{r} + \cos\theta\,\hat{\theta} .$$

$$\boxed{\dot{\hat{\phi}} = -\dot{\phi}\left(\sin\theta\,\hat{r} + \cos\theta\,\hat{\theta}\right)} .$$

$$\dot{\hat{\theta}} = \vec{\omega}\times\hat{\theta} = \dot{\theta}\hat{\phi}\times\hat{\theta} + \dot{\phi}\hat{k}\times\hat{\theta} .$$
$$= -\dot{\theta}\hat{r} + \dot{\phi}\cos\theta\,\hat{\phi}$$

$$\boxed{\dot{\hat{\theta}} = -\dot{\theta}\hat{r} + \dot{\phi}\cos\theta\,\hat{\phi}}$$

$$\ddot{\vec{r}} = R\ddot{\theta}\,\hat{\theta} - R\dot{\theta}^2\,\hat{r} + R\dot{\theta}\dot{\phi}\cos\theta\,\hat{\phi}$$
$$+\left(R\ddot{\phi}\sin\theta + R\dot{\phi}\dot{\theta}\cos\theta\right)\hat{\phi}$$
$$+\,R\dot{\phi}\sin\theta\left(-\dot{\phi}\sin\theta\,\hat{r} - \dot{\phi}\cos\theta\,\hat{\theta}\right).$$

$$\ddot{\vec{r}} = \left(-R\dot{\theta}^2 - R\dot{\phi}^2\sin^2\theta\right)\hat{r}$$
$$+\left(R\ddot{\theta} - R\dot{\phi}^2\sin\theta\cos\theta\right)\hat{\theta}$$
$$+\left(2R\dot{\phi}\dot{\theta}\cos\theta + R\ddot{\phi}\sin\theta\right)\hat{\phi}.$$

ya tenemos $\vec{r}$, $\dot{\vec{r}}$ y $\ddot{\vec{r}}$ en el marco de referencia rotatorio. luego:

$$\vec{\Omega} = \Omega(-\hat{k}) = -\Omega\left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right),$$

$$\vec{F} = M\vec{g} + \vec{N} + \vec{F}_{roce}$$

$$\vec{g} = +g\hat{k} = +g\left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right)$$

$$\vec{N} = N\hat{r}$$

$$\vec{F}_{roce} = -k\dot{\vec{r}} = -k\left(R\dot{\theta}\,\hat{\theta} + R\dot{\phi}\sin\theta\,\hat{\phi}\right).$$

A calcular :

$$\vec{\Omega} \times \vec{v}' = \vec{\Omega} \times \dot{\vec{r}} = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ -\Omega\cos\theta & \Omega\sin\theta & 0 \\ 0 & R\dot{\theta} & R\dot{\phi}\sin\theta \end{vmatrix}$$

$$= \begin{bmatrix} \Omega R\dot{\phi}\sin^2\theta \\ \Omega R\dot{\phi}\sin\theta\cos\theta \\ -\Omega R\dot{\theta}\cos\theta \end{bmatrix}$$

$$\vec{\Omega} \times \vec{r} = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ -\Omega\cos\theta & \Omega\sin\theta & 0 \\ R & 0 & 0 \end{vmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ -R\Omega\sin\theta \end{bmatrix}$$

$$\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ -\Omega\cos\theta & \Omega\sin\theta & 0 \\ 0 & 0 & -R\Omega\sin\theta \end{vmatrix}$$

$$= \begin{bmatrix} -R\Omega^2\sin^2\theta \\ -R\Omega^2\sin\theta\cos\theta \\ 0 \end{bmatrix}$$

De esta forma, la ecu. maestra queda escrita por componentes como:

$$\hat{r}: +Mg\cos\theta + N = -M\left(R\dot{\theta}^2 + R\dot{\phi}^2\sin^2\theta\right) + M\left[2\Omega R\dot{\phi}\sin^2\theta - R\Omega^2\sin^2\theta\right]$$

$$\hat{\theta}: -Mg\sin\theta - kR\dot{\theta} = M\left(R\ddot{\theta} - R\dot{\phi}^2\sin\theta\cos\theta\right) + M\left[2\Omega R\dot{\phi}\sin\theta\cos\theta - R\Omega^2\sin\theta\cos\theta\right]$$

$$\hat{\phi}: -kR\dot{\phi}\sin\theta = M\left(R\ddot{\phi}\sin\theta + 2R\dot{\theta}\dot{\phi}\cos\theta\right) + M\left[-2\Omega R\dot{\theta}\cos\theta\right]$$

En el equilibrio:

$$\dot{\theta} = \dot{\phi} = 0$$

$$\ddot{\theta} = \ddot{\phi} = 0$$

Luego:

$$\hat{\theta} : -Mg\sin\theta = -MR\Omega^2 \sin\theta\cos\theta$$

$$Mg\sin\theta\left[1 \; \text{[redacted]} \; \frac{R\Omega^2}{g}\cos\theta\right] = 0 \, .$$

$$\therefore \quad \boxed{\begin{aligned} \sin\theta &= 0 \\ \cos\theta &= \frac{g}{R\Omega^2} \, . \end{aligned}}$$

$\theta^* = 0$ es una posición de equilibrio

Si $\frac{g}{R\Omega^2} \le 1$ entonces $\theta^* = \cos^{-1}\left[\frac{g}{R\Omega^2}\right]$.

es otra posición de equilibrio .