P1

$z = x+iy = re^{i\theta} = r(\cos\theta + i\,\text{sen}\,\theta) = r\cos\theta + i\,r\,\text{sen}\,\theta$.

$\Rightarrow x = r\cos\theta$, $y = r\,\text{sen}\,\theta$; Además: $f(z) = \hat{u}(x,y) + i\hat{v}(x,y)$
$= u(r,\theta) + i v(r,\theta)$

Así: $u(r,\theta) = \hat{u}(r\cos\theta, r\,\text{sen}\,\theta)$
$v(r,\theta) = \hat{v}(r\cos\theta, r\,\text{sen}\,\theta)$

Como u y v son dif $\Rightarrow \hat{u}, \hat{v}$ también lo son (compos fn dif)

Luego, f es hol. en $\Omega \Leftrightarrow \underbrace{\hat{u}, \hat{v}}_{(u,v)}$ dif + CR. ✓

Así pues, basta escribir CR en (u,v) (r,θ) para concluir la equivalencia a holomorfía: (En realidad probaremos que (CR cartes) $\Rightarrow$ (CR polares) $\Leftarrow$ es análogo)

Por un lado en (x,y) (CR): $\partial_x \hat{u} = \partial_y \hat{v}$, $\partial_y \hat{u} = -\partial_x \hat{v}$

Calculemos derivadas de u, v c/r a (r,θ):

$$\partial_r u = \partial_r(\hat{u}(r\cos\theta, r\,\text{sen}\,\theta)) = \partial_x \hat{u} \cdot \partial_r(r\cos\theta) + \partial_y \hat{u} \cdot \partial_r(r\,\text{sen}\,\theta)$$
$$= \partial_x \hat{u} \cdot \cos\theta + \partial_y \hat{u} \cdot \text{sen}\,\theta$$

pero: $$\partial_\theta v = \partial_\theta(\hat{v}(r\cos\theta, r\,\text{sen}\,\theta)) = \partial_x \hat{v} \cdot \underbrace{\partial_\theta(r\cos\theta)} + \partial_y \hat{v} \cdot \underbrace{\partial_\theta(r\,\text{sen}\,\theta)}$$
$$= \partial_x \hat{v} \cdot -r\,\text{sen}\,\theta + \partial_y \hat{v} \cdot r\cos\theta.$$

pero por (CR - cartes): $\partial_x \hat{v} = -\partial_y \hat{u}$ y $\partial_y \hat{v} = \partial_x \hat{u}$
$$= -\partial_y \hat{u} \cdot (-r\,\text{sen}\,\theta) + \partial_x \hat{u} \cdot r\cos\theta$$

$$\therefore \frac{1}{r}\partial_\theta v = \frac{1}{\cancel{r}}(\partial_y \hat{u} \cdot \cancel{r}\,\text{sen}\,\theta + \partial_x \hat{u} \cdot \cancel{r}\cos\theta) = \cos\theta \cdot \partial_x \hat{u} + \text{sen}\,\theta\, \partial_y \hat{u}$$
$$\Rightarrow \boxed{\frac{1}{r}\partial_\theta v = \partial_r u}.$$

Análogamente se prueba la condición $\boxed{\frac{1}{r}\partial_\theta u = -\partial_r v}$

$\therefore$ Como $\hat{u}, \hat{v}$ $(\Leftrightarrow u, v)$ son dif + CR en pol $(\Leftrightarrow$ CR cart$)$ $\Leftrightarrow f$ hol. en Ω.

En el caso del logaritmo: $\text{Log } z = \log r + i\theta \quad (z = re^{i\theta})$

Veamos que es hol. con $r>0$ y $\theta \in (-\pi, \pi)$

Notar primero que $\log r = u(r, \theta) \quad \theta = v(r, \theta)$ son dif. en la región dada.

Obs. $v(r,\theta) = \theta$ solo en $(-\pi, \pi)$, $r>0$ si agrando el intervalo para θ $\Rightarrow v = \begin{cases} \theta - 2\pi \\ \theta \\ \theta + 2\pi \end{cases}$ pues es la función argumento! $\Rightarrow$ fuera de $(-\pi,\pi)$ No es dif

Luego, basta ver que cumplen las ecc. dadas:

$\partial_\theta v = 1, \quad \partial_r u = \frac{1}{r}, \quad \partial_\theta u = 0, \quad \partial_r v = 0.$

$\Rightarrow \frac{1}{r} \cdot \partial_\theta v = \frac{1}{r} \cdot 1 = \partial_r u$ ✓

$\frac{1}{r} \cdot \partial_\theta u = 0 = \partial_r v$ ✓

$\therefore$ log es holomorfo en $r>0$; $\theta \in (-\pi, \pi)$.

P2| $\lambda \in \mathbb{C}$. $p^\lambda : \mathbb{C}\setminus\{0\} \to \mathbb{C}$ vía: $p^\lambda(z) = \exp(\text{Log}(z)\cdot\lambda)$, $z \neq 0$.

a) Sea $k \in \mathbb{Z}$. Veamos que $p^k(z) = z^k$. Obs. Log denota al log complejo y log al usual.

$$\Rightarrow p^k(z) = \exp(k\,\text{Log}(z)) = \exp(k(\log|z| + i\theta)) \quad \theta \in (-\pi, \pi)$$

$$= \exp(k\log|z| + ik\theta) = \exp(\log|z|^k + ik\theta) = \exp(\log|z^k| + ik\theta)$$

$$\underset{z^k = |z^k| e^{ik\theta}}{=} \exp(\text{Log}(z^k)) \underset{\text{def}}{=} z^k.$$

Veamos además que $\forall \lambda \in \mathbb{C}$: $\overline{p^\lambda(\bar z)} = p^{\bar\lambda}(z)$:

Sea $\lambda = \lambda_1 + i\lambda_2 \in \mathbb{C}$. Además, notar que si $z = re^{i\theta} \Rightarrow \bar z = re^{-i\theta}$ $\Rightarrow \text{Log}(\bar z) = \log r - i\theta$.

Luego: $\overline{p^\lambda(\bar z)} = \overline{\exp(\lambda \text{Log } \bar z)} = \overline{\exp((\lambda_1 + i\lambda_2)(\log r - i\theta))}$

$$= \overline{\exp(\lambda_1 \log r - \lambda_1 i\theta + i\lambda_2 \log r + \lambda_2 \theta)} = \overline{\exp(\lambda_1 \log r + \lambda_2\theta)}\;\overline{\exp(i(\lambda_2 \log r - \lambda_1\theta))}$$

$$= \exp(\lambda_1 \log r + \lambda_2\theta)\exp(i(\lambda_1\theta - \lambda_2 \log r)) = \exp(\lambda_1 \log r + i\lambda_1\theta - i\lambda_2 \log r + \lambda_2\theta)$$

$$= \exp((\lambda_1 - i\lambda_2)(\log r + i\theta)) = \exp((\underbrace{\lambda_1 - i\lambda_2}_{\bar\lambda}) \operatorname{Log} z) = \exp(\bar\lambda \operatorname{Log} z)$$
$$= p^{\bar\lambda}(z) \checkmark$$

b) $\lambda, \mu \in \mathbb{C}$. Verif. que $p^{\lambda+\mu}(z) = p^{\lambda}(z) \cdot p^{\mu}(z)$.

$$p^{\lambda+\mu}(z) = \exp((\lambda+\mu) \operatorname{Log} z) = \exp(\lambda \operatorname{Log} z + \mu \operatorname{Log}(z))$$
$$= \exp(\lambda \operatorname{Log}(z)) \exp(\mu \operatorname{Log}(z)) = p^{\lambda}(z) \cdot p^{\mu}(z).$$

¿Dónde es hol $p^{\lambda}(z)$?

Basta ver que p^{λ} es compos de $\exp(z)$ hol. en todo $\mathbb{C}$

y $\operatorname{Log}(z)$: hol. en $\mathbb{C} \setminus \mathbb{R}_-$

$\Rightarrow p^{\lambda}$ es hol en $\mathbb{C} \setminus \mathbb{R}_-$.

Probemos que $(p^{\lambda})'(z) = \lambda p^{\lambda-1}(z)$

$$(p^{\lambda})' = (\exp(\operatorname{Log}(z) \cdot \lambda))' = \exp(\operatorname{Log}(z) \cdot \lambda) \cdot (\operatorname{Log}(z) \cdot \lambda)'$$
$$= \exp(\operatorname{Log}(z) \cdot \lambda) \cdot z^{-1} \cdot \lambda$$

pero $z^{-1} = \exp(-\operatorname{Log}(z))$ ($-1 \in \mathbb{Z}$)

$$= \exp(\operatorname{Log}(z) \cdot \lambda) \exp(-\operatorname{Log}(z)) \cdot \lambda$$
$$= \exp((\lambda - 1)\operatorname{Log}(z)) \cdot \lambda$$
$$= p^{\lambda-1}(z) \cdot \lambda \checkmark$$

c) Pdq: Dados $\alpha, \beta \in \mathbb{R}$, $t > 0$: $t^{\alpha+i\beta} = t^{\alpha}[\cos(\beta \log t) + i \sin(\beta \log t)]$

En efecto: $t^{\alpha+i\beta} = t^{\alpha} t^{i\beta} = t^{\alpha}(\exp(i\beta \log t)) \underset{t \in \mathbb{R}^+}{=} t^{\alpha}(\exp(i\underbrace{\beta \log t}_{\theta \in \mathbb{R}}))$

$$= t^{\alpha}(\cos(\theta) + i \operatorname{sen}(\theta)) = t^{\alpha}(\cos(\beta \log t) + i \operatorname{sen}(\beta \log t)) \checkmark$$

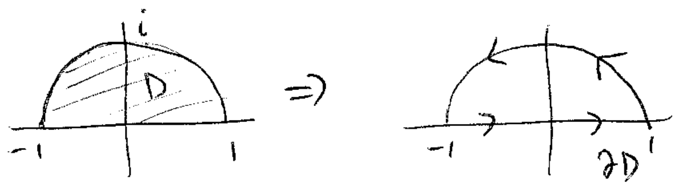
Finalmente: $i^i = \exp(i \operatorname{Log}(i)) = \exp(i(\underbrace{\log|i|}_{0} + i\frac{\pi}{2})) = \exp(+i^2 \frac{\pi}{2}) = \exp(-\pi/2)$

$$= e^{-\pi/2} \in \mathbb{R}!$$

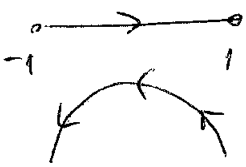
P3 a) $\oint_\Gamma |z|\bar{z}\,dz$ con $\Gamma = \partial\{z \in \mathbb{C} \,/\, |z| \leq 1 \wedge \operatorname{Im} z \geq 0\} = \partial D$

recordemos que si $\gamma(t) : [a,b] \to \mathbb{C}$ param a Γ:

$$\oint_\Gamma f(z)dz = \int_a^b f(\gamma(t))\cdot\gamma'(t)\,dt$$



$\partial D = \Gamma_1 \cup \Gamma_2$ con Γ_1 :



Param. de Γ_1: $\gamma_1(t) = t \quad t \in [-1,1] \Rightarrow \gamma_1'(t) = 1$

$$\Rightarrow \int_{\Gamma_1} |z|\bar{z}\,dz = \int_{-1}^{1} |t|\cdot t\,dt = 0 \quad (|t|\cdot t \text{ es impar})$$

Param. de Γ_2: $\gamma_2(\theta) = e^{i\theta} \quad \theta \in [0,\pi] \Rightarrow \gamma_2'(\theta) = ie^{i\theta}$

$$\Rightarrow \int_{\Gamma_2} |z|\bar{z}\,dz = \int_0^\pi \underbrace{|e^{i\theta}|}_{1}\cdot e^{-i\theta}\cdot ie^{i\theta}\,d\theta = \int_0^\pi i\,d\theta = i\pi$$

Finalmente: $\oint_\Gamma |z|\bar{z}\,dz = \int_{\Gamma_1} + \int_{\Gamma_2} = 0 + i\pi = i\pi$.

[redacted]

b) Pdq: Si $\Gamma \subset \mathbb{C}$ cerrado simple, recorr. antihor. y que encierra a D

$$\Rightarrow \text{Área}(D) = \frac{1}{2i}\oint_\Gamma \bar{z}\,dz$$

Sol: Para resolver este problema, escribiremos (por única vez fuera de la dem. de Cauchy-Goursat) la integral compleja como integrales en $\mathbb{R}^2$, para usar el Teo. de Green de forma apropiada.

notar que: $\frac{1}{2i}\oint_\Gamma \bar{z}\,dz = \frac{1}{2i}\oint_\Gamma (x+iy)(dx+idy) = \frac{1}{2i}\oint_\Gamma (xdx + ixdy + iydx + \underbrace{i^2}_{-1}ydy)$

$= \frac{1}{2i}\left(\oint_\Gamma xdx - ydy + i\oint_\Gamma xdy + ydx\right)$ pero Green dice que $\oint_{\Gamma=\partial D} Mdx + Ndy = \iint_D \left(\frac{\partial M}{\partial y} + \frac{\partial N}{\partial x}\right)dxdy$

$\Rightarrow = \frac{1}{2i}\left(\iint_D \left(\frac{\partial x}{\partial y} - \frac{\partial y}{\partial x}\right)dxdy + i\iint_D\left(\frac{\partial y}{\partial y} + \frac{\partial x}{\partial x}\right)dxdy\right) = \frac{1}{2i}\cdot(0 + 2i\,\text{Área}(D)) = \text{Área}(D)$ $\square$

P4 a) $f \longmapsto \frac{f'}{f}$ envía prod. en sumas (f Hol!)

En efecto: $f\cdot g \longmapsto \frac{(f\cdot g)'}{f\cdot g} = \frac{f'g + fg'}{fg} = \frac{f'\cancel{g}}{f\cancel{g}} + \frac{\cancel{f}g'}{\cancel{f}g} = \frac{f'}{f} + \frac{g'}{g}$.

Tal como se deseaba.

b) $P(z) = (z-a_1)(z-a_2)\cdots(z-a_n) \quad a_i \in \mathbb{C}$.

$$\frac{P'(z)}{P(z)} = \frac{\left((z-a_1)\cdot((z-a_2)\cdots(z-a_n))\right)'}{(z-a_1)\cdots(z-a_n)} \underset{\substack{\uparrow \\ a)}}{=} \frac{(z-a_1)'}{(z-a_1)} + \frac{((z-a_2)\cdots(z-a_n))'}{(z-a_2)\cdots(z-a_n)}$$

$$= \frac{1}{z-a_1} + \frac{\left((z-a_2)\cdot((z-a_3)\cdots(z-a_n))\right)'}{\underbrace{(z-a_2)\cdot((z-a_3)\cdots(z-a_n))}_{\text{prod de nuevo, puedo iterar esto } n-2 \text{ veces}}} = \frac{1}{z-a_1} + \frac{1}{z-a_2} + \cdots + \frac{1}{z-a_n}$$

c) γ cerrada que <u>no</u> pasa por raíces de P (esto hace que la integral esté bien definida)

<u>Pdq</u> $\frac{1}{2\pi i}\oint_\gamma \frac{P'(z)}{P(z)}\,dz = \sum_{i=1}^{n} \mathrm{Ind}(\gamma, a_i)$

<u>Sol.</u> Recordemos que, por definición $\mathrm{Ind}(\gamma, p) = \frac{1}{2\pi i}\oint_\gamma \frac{1}{z-p}\,dz$

y por las partes anteriores:

$$\frac{1}{2\pi i}\oint_\gamma \frac{P'(z)}{P(z)}\,dz = \frac{1}{2\pi i}\oint_\gamma \left(\frac{1}{z-a_1} + \frac{1}{z-a_2} + \cdots + \frac{1}{z-a_n}\right) dz$$

$$\underset{\substack{\uparrow \\ \text{lin integ}}}{=} \frac{1}{2\pi i}\oint_\gamma \frac{1}{z-a_1}\,dz + \frac{1}{2\pi i}\oint_\gamma \frac{1}{z-a_2}\,dz + \cdots + \frac{1}{2\pi i}\oint_\gamma \frac{1}{z-a_n}\,dz = \sum_{i=1}^{n} \mathrm{Ind}(\gamma, a_i)$$

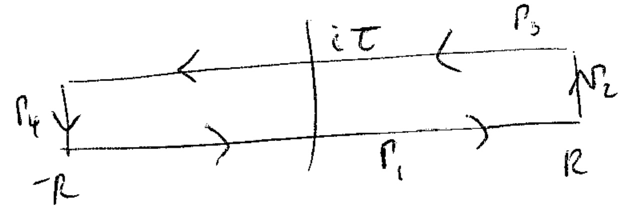
como se deseaba.

P5| Pdq. $\int_{\mathbb{R}} e^{-x^2}\cos(2bx)\,dx = e^{-b^2}\frac{\sqrt{\pi}}{2} \quad \forall b \in \mathbb{R}$

Sol. Notar que el caso $b=0$ es conocido $\left(\int_{\mathbb{R}} e^{-x^2}dx = \frac{\sqrt{\pi}}{2} \right)$

Sea $b \neq 0$. Consideremos $f(z) = e^{-z^2}e^{2ibz}$ que es holomorfa en todo $\mathbb{C}$ al ser compos. de Holomorfas.

Consid. el camino sgte:



Por Cauchy-Goursat (o argumentando con series de potencia) (detallado en clase)

$$\oint_{\Gamma} f(z)dz = 0 = \sum_{i=1}^{4}\int_{\Gamma_i} f(z)dz \quad \forall R>0 \quad \forall \tau > 0$$

$$\Gamma = \bigcup_{i=1}^{4}\Gamma_i$$

Calculemos pues las integrales y luego veamos el límite con $R \to \infty$

Γ1| Param: $\gamma_1(t) = t, \ t \in [-R,R] \Rightarrow \gamma_1'(t) = 1$

$$\Rightarrow \int_{\Gamma_1} f(z)dz = \int_{-R}^{R} e^{-t^2}e^{2ibt}\cdot 1\,dt = \int_{-R}^{R} e^{-t^2}(\cos(2bt) + i\,\mathrm{sen}(2bt))dt$$

$$= \int_{-R}^{R} e^{-t^2}\cos(2bt)dt + i\underbrace{\int_{-R}^{R} e^{-t^2}\mathrm{sen}(2bt)dt}_{=0 \text{ por paridad}}$$

Γ3| Param: $\gamma_3(t) = -t+i\tau, \ t \in [-R,R] \Rightarrow \gamma_3'(t) = -1$.

$$\Rightarrow \int_{\Gamma_3} f(z)dz = \int_{-R}^{R} e^{-(i\tau-t)^2} e^{+2ib(i\tau-t)}\cdot(-1)dt = -\int_{-R}^{R} e^{-t^2}e^{\tau^2}e^{2i\tau t}e^{-2ibt}e^{-2b\tau}dt$$

$$= -e^{\tau^2-2b\tau}\int_{-R}^{R} e^{-t^2}e^{i(2\tau t-2bt)}dt = -e^{+\tau^2-2b\tau}\int_{-R}^{R} e^{-t^2}e^{i2t(\tau-b)}dt$$

Notar que en este punto es posible afirmar que si:

$$\lim_{R\to\infty}\int_{\Gamma_2} f(z)dz = \lim_{R\to\infty}\int_{\Gamma_4} f(z)dz = 0 \qquad (*)$$

Entonces: $\lim_{R\to\infty}\oint_{\Gamma} f(z)dz = 0 = \lim_{R\to\infty}\int_{\Gamma_1} f(z)dz + \lim_{R\to\infty}\int_{\Gamma_3} f(z)dz$

$$= \int_{\mathbb{R}} e^{-t^2}\cos(2bt)\,dt + -e^{\tau^2-2b\tau}\int_{\mathbb{R}} e^{-t^2}e^{it(\tau-b)\cdot 2}dt$$

$$\Rightarrow \int_{\mathbb{R}} e^{-t^2}\cos(2bt)\,dt = e^{\tau^2-2b\tau}\int_{\mathbb{R}} e^{-t^2}e^{i2t(\tau-b)}dt \quad \forall \tau>0$$

Tomando $\tau=b>0$

$$\Rightarrow \int_{\mathbb{R}} e^{-t^2}\cos(2bt)\,dt = e^{b^2-2b^2}\int_{\mathbb{R}} e^{-t^2}\cdot \underbrace{e^{i2t\cdot 0}}_{1}dt$$

$$= e^{-b^2}\int_{\mathbb{R}} e^{-t^2}dt = e^{-b^2}\frac{\sqrt{\pi}}{2}$$ que es el resultado deseado.

Así pues, para concluir basta demostrar que se cumple $(*)$.

En efecto: para $\underline{\Gamma_2}$ $\gamma_2(t)=R+it \Rightarrow \gamma_2'(t)=i \qquad t\in[0,\tau]$

$$\Rightarrow \left|\int_{\Gamma_2} f(z)dz\right| \leq \int_{\Gamma_2}|f(z)||dz| = \int_0^\tau |f(R+it)||i|dt = \int_0^\tau \left|e^{-(R+it)^2}e^{2bi(R+it)}\right|\cdot dt$$

$$= \int_0^\tau \left|e^{-R^2-2Rit+t^2}e^{i(2bR+i2bt)}\right|dt = e^{-R^2}\int_0^\tau e^{t^2}\cdot e^{-2bt}\cdot\underbrace{\left|e^{i\underbrace{(2bR-2Rt)}_{\in\mathbb{R}}}\right|}_{=1}dt$$

$$= e^{-R^2}\int_0^\tau e^{t^2}\cdot e^{-2bt}dt \leq e^{-R^2}\cdot\underbrace{\sup_{t\in[0,\tau]} e^{t^2}e^{-2bt}}_{=M<\infty \text{ pues la función es continua}}\cdot\tau \xrightarrow[R\to\infty]{} 0$$

$$\therefore \lim_{R\to\infty} \int_{\Gamma_2} f(z)dz = 0.$$

$\underline{\Gamma_4}$ $i\tau - R \to -R$ $\quad \gamma_4(t) = -R + i(\tau - t) \quad t\in[0,\tau]$

$\gamma_4'(t) = -i$

$$\Rightarrow \left|\int_{\Gamma_4} f(z)dz\right| = \left|\int_0^\tau f(\gamma(t))\cdot\gamma'(t)\,dt\right| \leq \int_0^\tau |f(\gamma(t))|\cdot\underbrace{|\gamma'(t)|}_{=|-i|=1}dt$$

$$\leq \int_0^\tau \left|e^{-(-R+i(\tau-t))^2} e^{2bi(i(\tau-t)-R)}\right| dt$$

$$= \int_0^\tau \left|e^{-(R^2-2Ri(\tau-t)-(\tau-t)^2)} e^{-2bRi} e^{2b(t-\tau)}\right| dt$$

$$= e^{-R^2}\int_0^\tau e^{-(\tau-t)^2}\cdot\underbrace{\left|e^{i\overbrace{(2R(\tau-t) - 2bR)}^{\alpha\in\mathbb{R}}}\right|}_{1}\cdot e^{2b(\tau-t)}\,dt$$

$$= e^{-R^2}\int_0^\tau e^{-(\tau-t)^2} e^{2b(\tau-t)}\,dt$$

$$\leq e^{-R^2}\cdot\underbrace{\sup_{t\in[0,\tau]} e^{-(\tau-t)^2+2b(\tau-t)}}_{=:M,\text{ finito pues es func cont en un compacto}}\cdot\tau \leq e^{-R^2}\tau\cdot M \xrightarrow[R\to\infty]{} 0$$

$$\therefore \lim_{R\to\infty}\int_{\Gamma_4} f(z)dz = 0$$

y se concluye pues hemos probado $(*)$ $\square$