P1) $\vec{F}: \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}^3$, $f, g: \Omega \to \mathbb{R}$

a) $\nabla(f \cdot g) = f \cdot \nabla g + g \nabla f$

Sean $x_1 = x$, $x_2 = y$, $x_3 = z$; $\hat{e}_1 = \hat{\imath}$, $\hat{e}_2 = \hat{\jmath}$, $\hat{e}_3 = \hat{k}$

$$\Rightarrow \nabla(f \cdot g) = \sum_{i=1}^{3} \frac{\partial(f \cdot g)}{\partial x_i} \hat{e}_i = \sum_{i=1}^{3} \left(\frac{\partial f}{\partial x_i} g + f \frac{\partial g}{\partial x_i} \right) \hat{e}_i$$

$$= g \cdot \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} \hat{e}_i + f \sum_{i=1}^{3} \frac{\partial g}{\partial x_i} \hat{e}_i$$

$$= g \cdot \nabla f + f \cdot \nabla g \quad \checkmark$$

b) $\operatorname{div}(f \cdot \vec{F}) = \nabla f \circ \vec{F} + f \operatorname{div}(\vec{F})$

notar primero que $f\vec{F} = f(F_1, F_2, F_3)$ $\quad F_i : \Omega \to \mathbb{R}$.

$= (fF_1, fF_2, fF_3)$ $\quad (F_i = \vec{F} \cdot \hat{e}_i)$

$$\therefore \operatorname{div}(f \cdot \vec{F}) = \sum_{i=1}^{3} \frac{\partial (fF_i)}{\partial x_i} = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} F_i + \sum_{i=1}^{3} f \frac{\partial F_i}{\partial x_i}$$

$$= \nabla f \circ \vec{F} + f \operatorname{div}(\vec{F}) \quad \checkmark$$

c) $\operatorname{rot}(f\vec{F}) = \nabla f \times \vec{F} + f \operatorname{rot} \vec{F}$.

$$\operatorname{rot}(f\vec{F}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ fF_1 & fF_2 & fF_3 \end{vmatrix} = \hat{\imath}\left(\partial_y (fF_3) - \partial_z (fF_2)\right) + \hat{\jmath}\left(\partial_z (fF_1) - \partial_x (fF_3)\right) + \hat{k}\left(\partial_x (fF_2) - \partial_y (fF_1)\right)$$

$$= \hat{\imath}\left(\partial_y f \cdot F_3 - \partial_z f \cdot F_2\right) + \hat{\jmath}\left(\partial_z f \cdot F_1 - \partial_x f \cdot F_3\right) + \hat{k}\left(\partial_x f \cdot F_2 - \partial_y f \, F_1\right)$$
$$+ \hat{\imath}\left(f\left(\partial_y F_3 - \partial_z F_2\right)\right) + \hat{\jmath}\left(f \cdot \left(\partial_z F_1 - \partial_x F_3\right)\right) + \hat{k}\left(f\left(\partial_x F_2 - \partial_y F_1\right)\right)$$
$$= \nabla f \times \vec{F} + f \operatorname{rot} \vec{F}$$

P2) a) $f(x,y,z) = \dfrac{\arccos\left(\dfrac{z}{\sqrt{x^2+y^2+z^2}}\right)}{x^2+y^2+z^2}$

Calculemos ∇f: 2 opciones:

1) Mala opción: $\nabla f(x,y,z) = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i}\hat{e}_i$ Calculos gigantes!

2) Usar coordenadas ad-hoc a la simetría de la función:
Notar que en esféricas: $x = r\sin\varphi\cos\theta$, $y = r\sin\varphi\sin\theta$, $z = r\cos\varphi$

$$f(r,\theta,\varphi) = \frac{\arccos\left(\frac{r\cos\varphi}{r}\right)}{r^2} = \frac{\varphi}{r^2}$$ muy simple y fácil para derivar!

En esféricas: $h_r = 1$, $h_\theta = r\sin\varphi$, $h_\varphi = r$

$$\nabla f(r,\theta,\varphi) = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r\sin\varphi}\frac{\partial f}{\partial\theta}\hat{\theta} + \frac{1}{r}\frac{\partial f}{\partial\varphi}\hat{\varphi}$$

$$\therefore \nabla f(r,\theta,\varphi) = \frac{\partial}{\partial r}\left(\frac{\varphi}{r}\right)\hat{r} + \cancel{\frac{1}{r\sin\varphi}\frac{\partial}{\partial\theta}\left(\frac{\varphi}{r}\right)\hat{\theta}}^{\,0} + \frac{1}{r}\frac{\partial}{\partial\varphi}\left(\frac{\varphi}{r}\right)\hat{\varphi}$$

$$= -\frac{\varphi}{r^2}\hat{r} + \frac{1}{r^2}\hat{\varphi} = \frac{1}{r^2}\left(\hat{\varphi} - \varphi\hat{r}\right)$$

b) $\varepsilon > 0$, $\eta > 0$, $\phi \in [0, 2\pi]$. Queremos $\nabla f(\varepsilon,\eta,\phi)$ y $\Delta f(\varepsilon,\eta,\phi)$.

$$\vec{r}(\varepsilon,\eta,\phi) = \left(\varepsilon\eta\cos\phi,\ \varepsilon\eta\sin\phi,\ \tfrac{1}{2}(\eta^2-\varepsilon^2)\right)$$

Calculemos los factores escalares:

$$\frac{\partial\vec{r}}{\partial\varepsilon} = (\eta\cos\phi, \eta\sin\phi, -\varepsilon) \Rightarrow h_\varepsilon = \left\|\frac{\partial\vec{r}}{\partial\varepsilon}\right\| = \sqrt{\eta^2+\varepsilon^2}$$

$$\frac{\partial \vec{r}}{\partial \eta} = (\varepsilon\cos\phi, \varepsilon \operatorname{sen}\phi, -\eta) \Rightarrow h_\eta = \left\|\frac{\partial \vec{r}}{\partial \eta}\right\| = \sqrt{\varepsilon^2+\eta^2}$$

$$\frac{\partial \vec{r}}{\partial \phi} = (-\varepsilon\eta \operatorname{sen}\phi, \varepsilon\eta\cos\phi, 0) \Rightarrow h_\phi = \left\|\frac{\partial \vec{r}}{\partial \phi}\right\| = \sqrt{\varepsilon^2\eta^2} = |\varepsilon\eta| \underset{\varepsilon,\eta\geq 0}{=} \varepsilon\eta.$$

$\therefore$ $h_\varepsilon = h_\eta = \sqrt{\varepsilon^2+\eta^2}$ $\quad h_\phi = \varepsilon\eta$

$$\therefore \quad \nabla f(\varepsilon,\eta,\phi) = \frac{1}{h_\varepsilon}\frac{\partial f}{\partial \varepsilon}\hat{\varepsilon} + \frac{1}{h_\eta}\frac{\partial f}{\partial \eta}\hat{\eta} + \frac{1}{h_\phi}\frac{\partial f}{\partial \phi}\hat{\phi}$$

$$\boxed{\nabla f(\varepsilon,\eta,\phi) = \frac{1}{\sqrt{\varepsilon^2+\eta^2}}\frac{\partial f}{\partial \varepsilon}\hat{\varepsilon} + \frac{1}{\sqrt{\varepsilon^2+\eta^2}}\frac{\partial f}{\partial \eta}\hat{\eta} + \frac{1}{\varepsilon\eta}\frac{\partial f}{\partial \phi}\hat{\phi}}$$

Para el Laplaciano, recordemos que, dada $f: \Omega \subset \mathbb{R}^3 \to \mathbb{R}$:

$$\Delta f = div(\nabla f)$$ → ya lo tenemos en estas coord.

En gral: $$div(\vec{F}) = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}(F_u h_v h_w) + \frac{\partial}{\partial v}(F_v h_u h_w) + \frac{\partial}{\partial w}(F_w h_u h_v)\right]$$

En nuestro caso: $u = \varepsilon$, $v = \eta$, $w = \phi$

$$\Rightarrow F_u = \nabla f \cdot \hat{u} = \nabla f \cdot \hat{\varepsilon} = \frac{1}{h_\varepsilon}\frac{\partial f}{\partial \varepsilon}, \quad F_v = \frac{1}{h_\eta}\frac{\partial f}{\partial \eta}, \quad F_w = \frac{1}{h_\phi}\frac{\partial f}{\partial \phi}$$

$$\therefore \quad \Delta f(\varepsilon,\eta,\phi) = div(\nabla f(\varepsilon,\eta,\phi))$$

$$= \frac{1}{\varepsilon\eta(\varepsilon^2+\eta^2)}\left[\frac{\partial}{\partial \varepsilon}\left(\frac{1}{\cancel{h_\varepsilon}}\frac{\partial f}{\partial \varepsilon}\cdot \cancel{h_\eta} h_\phi\right) + \frac{\partial}{\partial \eta}\left(\frac{1}{\cancel{h_\eta}}\frac{\partial f}{\partial \eta}\cancel{h_\varepsilon} h_\phi\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{h_\phi}\frac{\partial f}{\partial \phi}h_\varepsilon h_\eta\right)\right]$$

$$\boxed{\Delta f(\varepsilon,\eta,\phi) = \frac{1}{\varepsilon\eta(\varepsilon^2+\eta^2)}\left[\frac{\partial}{\partial \varepsilon}\left(\frac{\partial f}{\partial \varepsilon}\cdot\varepsilon\eta\right) + \frac{\partial}{\partial \eta}\left(\frac{\partial f}{\partial \eta}\varepsilon\eta\right) + \frac{\partial}{\partial \phi}\left(\frac{\varepsilon^2+\eta^2}{\varepsilon\eta}\frac{\partial f}{\partial \phi}\right)\right]}$$

P4 Ecc. Euler: $\rho \nabla\vec{v}\,\vec{v} + \nabla p = -\rho g \hat{k}$

$\vec{v}$: Campo vect de velocidades
ρ: densidad
p: campo esc. de presiones.

$\vec{v} = (v_1, v_2, v_3)$ $\quad p = p(x,y,z)$.

Obs: Dado $\vec{v}: \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ entendemos

$$\nabla\vec{v}\cdot\vec{v} = \begin{pmatrix} \sum v_i \partial x_i v_1 \\ \sum v_i \partial x_i v_2 \\ \sum v_i \partial x_i v_3 \end{pmatrix} \overset{Not.}{=} (\underbrace{\vec{v}\cdot\nabla}_{\sum v_i \partial x_i})\vec{v}$$

a) Probemos que $\nabla\vec{v}\cdot\vec{v} = \frac{1}{2}\nabla(v_1^2+v_2^2+v_3^2) - \vec{v}\times(\nabla\times\vec{v})$.

$$\frac{1}{2}\nabla(v_1^2+v_2^2+v_3^2) = \frac{1}{2}\begin{pmatrix} \partial_x(v_1^2+v_2^2+v_3^2) \\ \partial_y(\quad \text{"} \quad) \\ \partial_z(\quad \text{"} \quad) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2(v_1\partial_x v_1 + v_2\partial_x v_2 + v_3\partial_x v_3) \\ 2(v_1\partial_y v_1 + v_2\partial_y v_2) + \dots \\ 2(v_1\partial_z v_1 + v_2\partial_z v_2 + v_3\partial_z v_3) \end{pmatrix}$$

$$= \begin{pmatrix} \sum v_i \partial_x v_i \\ \sum v_i \partial_y v_i \\ \sum v_i \partial_z v_i \end{pmatrix}$$

Por otro lado: $\vec{v}\times(\nabla\times\vec{v}) = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \times \begin{pmatrix} \partial_y v_3 - \partial_z v_2 \\ \partial_z v_1 - \partial_x v_3 \\ \partial_x v_2 - \partial_y v_1 \end{pmatrix}$

Rcdo: $\hat{\imath} \overset{+}{\to} \hat{\jmath} \overset{+}{\to} \hat{k}$, $\hat{k} \overset{+}{\to} \hat{\imath}$

(i.e. $\hat{\imath}\times\hat{\jmath} = \hat{k}$, $\hat{\jmath}\times\hat{\imath} = -\hat{k}$)

$$= \begin{pmatrix} v_2\partial_x v_2 - v_2\partial_y v_1 - v_3\partial_z v_1 + v_3\partial_x v_3 \\ v_1\partial_y v_1 - v_1\partial_x v_2 + v_3\partial_y v_3 - v_3\partial_z v_2 \\ \end{pmatrix}$$

$$\Rightarrow \frac{1}{2}\nabla(v_1^2+v_2^2+v_3^2) - \vec{v}\times(\nabla\times\vec{v}) = \begin{pmatrix} v_1\partial_x v_1 + v_2\partial_y v_1 + v_3\partial_z v_1 \\ v_1\partial_x v_2 + v_2\partial_y v_2 + v_3\partial_z v_2 \\ v_1\partial_x v_3 + v_2\partial_y v_3 + v_3\partial_z v_3 \end{pmatrix}$$

$$= (\vec{v}\cdot\nabla)\vec{v}$$

b) ρ cte + $\nabla\times\vec{v} = 0$

Ecc. Euler: $\rho\,\nabla\vec{v}\cdot\vec{v} + \nabla p = -\rho g\hat{k}$

por a)

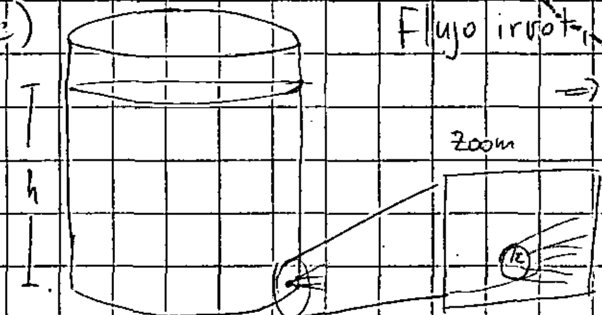
$$\underbrace{\rho}_{cte}\left(\frac{1}{2}\nabla(v_1^2+v_2^2+v_3^2) - \vec{v}\times\underbrace{(\nabla\times\vec{v})}_{0}\right) + \nabla p = -\rho g\hat{k}$$

$$\nabla\left(\frac{\rho}{2}(v_1^2+v_2^2+v_3^2)\right) + \nabla p = \nabla(-\rho g z)$$

∇ es lineal! $\quad \nabla\left(\frac{\rho}{2}(v_1^2+v_2^2+v_3^2) + p + \rho g z\right) = 0.$

$$\Rightarrow \frac{\rho}{2}(v_1^2+v_2^2+v_3^2) + p + \rho g z = cte. \quad \forall (x,y,z)\in\Omega.$$

c)



Flujo irrot., incomp. y estacionario (h cte).

$\Rightarrow$ vale b), ie. $\forall (x,y,z) \in$ cil

$$\frac{\rho}{2}(v_1^2+v_2^2+v_3^2) + p + \rho g z = cte.$$

Pdq $v_{salida} = \sqrt{2gh}$

Aprox 1: h cte en el tpo (basta asumir que $h \gg \varepsilon$)

Aprox 2: La vel de salida es solo horizontal

En $z=h$: $v_1^2+v_2^2+v_3^2 = 0$. $p = p_{atm} \Rightarrow \rho g h = cte$

En la salida: $\frac{\rho}{2}v^2 + p_{atm} + 0 = cte$

$$\Rightarrow \frac{\rho}{2}v^2 + p_{atm} = \rho g h + p_{atm} \Rightarrow \boxed{v = \sqrt{2gh}}$$

P3) $\vec{F}(\vec{r}) = F_\rho(\rho)\hat{\rho}$, $\rho > 0$, $F_\rho : (0,\infty)\to\mathbb{R}$, $F_\rho \in C^1$

a) $\mathrm{rot}(F_\rho(\rho)\hat{\rho}) = 0$. Recordemos que, en gral:

$$\mathrm{rot}(\vec{F}(\vec{r})) = \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u\hat{u} & h_v\hat{v} & h_w\hat{w} \\ \partial_u & \partial_v & \partial_w \\ h_u F_u & h_v F_v & h_w F_w \end{vmatrix}$$

Obs. La expr con det vale si los vect unit están ordenados según mano derecha.

En nuestro caso (cilíndricas) $u=\rho \quad v=\theta \quad w=z$

$\Rightarrow h_u = 1 \quad h_v = \rho \quad h_w = 1$

$\therefore \quad rot(\vec{F}) = \dfrac{1}{1\cdot\rho\cdot 1} \begin{vmatrix} \hat{\rho} & \rho\hat{\theta} & \hat{k} \\ \partial_\rho & \partial_\theta & \partial_z \\ F_\rho & \rho\cdot F_\theta & F_z \end{vmatrix}$ y en nuestro caso, $F_\theta = F_z = 0$

$\Rightarrow \quad rot(F_\rho\hat{\rho}) = \dfrac{1}{\rho} \begin{vmatrix} \hat{\rho} & \rho\hat{\theta} & \hat{k} \\ \partial_\rho & \partial_\theta & \partial_z \\ F_\rho & 0 & 0 \end{vmatrix} = \dfrac{1}{\rho}\cdot\left[\hat{\rho}(0-0) + \rho\hat{\theta}(\overset{0}{\cancel{\partial_z F_\rho}} - 0) + \hat{k}(0 - \overset{0}{\cancel{\partial_\theta F_\rho}})\right]$

$= 0$ como se quería

Obs. Estas operaciones valen en $\mathbb{R}^3 \setminus \{eje\ z\}$ pues tenemos el término $\frac{1}{\rho}$!!

b) $div(F_\rho(\rho)\hat{\rho}) = \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}(F_\rho\,\rho)$

En efecto, como en gral $div\,\vec{F} = \dfrac{1}{h_u h_v h_w}\left[\dfrac{\partial}{\partial u}(F_u h_v h_w) + \dfrac{\partial}{\partial v}(F_v h_u h_w) + \dfrac{\partial}{\partial w}(F_w h_u h_v)\right]$

en nuestro caso: $u=\rho,\ v=\theta,\ w=z;\ F_u = F_\rho,\ F_v = F_w = 0$

y $h_u = h_\rho = 1,\ h_v = h_\theta = \rho,\ h_w = h_z = 1$

$\Rightarrow div\,\vec{F} = div\left(F_\rho(\rho)\hat{\rho}\right) = \dfrac{1}{\rho}\left[\dfrac{\partial}{\partial\rho}(F_\rho\cdot\rho) + 0 + 0\right]$

$= \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}(F_\rho\cdot\rho)$ ✓

c) $div(\underbrace{F_\rho(\rho)\hat{\rho}}_{\vec{F}(\vec{r})}) = 0 \iff \vec{F}(\vec{r}) = K/\rho\ \hat{\rho}$

$\Leftarrow$) Obvio: $div\left(K/\rho\ \hat{\rho}\right) = \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\dfrac{K}{\rho}\cdot\rho\right) = \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}(K) = 0$

$\Rightarrow$) $div(F_\rho\hat{\rho}) = 0 \iff \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}(F_\rho\cdot\rho) = 0$ en $\mathbb{R}^3 \setminus \{eje\ z\}$

$\Rightarrow \dfrac{\partial}{\partial\rho}(F_\rho\cdot\rho) = 0 \iff \rho\cdot F_\rho' + F_\rho = 0 \iff F_\rho' = -\dfrac{F_\rho}{\rho}$

$\dfrac{F_\rho'}{F_\rho} = -\dfrac{1}{\rho} \Rightarrow \ln(F_\rho) = -\ln(\rho) + C \Rightarrow \boxed{F_\rho = \dfrac{K}{\rho}}$