Auxiliar 6 - CW - 2012/2

P1) a) $f(x) = x^T A x \quad f: \mathbb{R}^N \to \mathbb{R}$. $A \in \mathcal{M}_{N\times N}(\mathbb{R})$, A simet.

Pdq: $\nabla f(x) = 2(Ax)^T$.

Sol. Escribamos un "poco mejor" f, para tener una expresión más amigable: $x = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}$, $A = \begin{pmatrix} A_1 \\ \vdots \\ A_N \end{pmatrix}$ con $A_i = (a_{i1}\ a_{i2}\ \dots\ a_{iN})$

$$\Rightarrow f(x) = x^T A x = (x_1 \dots x_N) \underbrace{\begin{pmatrix} A_1 \\ \vdots \\ A_N \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}}_{\begin{pmatrix} A_1 \cdot x \\ A_2 \cdot x \\ \vdots \\ A_N \cdot x \end{pmatrix}} \text{ (prod pto)}$$

$$= (x_1 \dots x_N)\begin{pmatrix} A_1 \cdot x \\ \vdots \\ A_N \cdot x \end{pmatrix} = \sum_{i=1}^N x_i (A_i \cdot x)$$

$$= \sum_{i=1}^N x_i \cdot \sum_{j=1}^N a_{ij} \cdot x_j$$

Con esto es fácil ver que f es $C^\infty(\mathbb{R}^N, \mathbb{R})$ (compos. de polinomios).

Para calcular $\frac{\partial f}{\partial x_\ell}$ $\ell \in \{1, \dots, N\}$ (y así tener ∇f) usaremos la notación del delta de Kroenecker: $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

$$\Rightarrow \frac{\partial f}{\partial x_\ell} = \frac{\partial}{\partial x_\ell}\left(\sum_{i=1}^N x_i \cdot \sum_{j=1}^N a_{ij} \cdot x_j\right) = \sum_{i=1}^N \frac{\partial x_i}{\partial x_\ell} \cdot \sum_{j=1}^N a_{ij} \cdot x_j + \sum_{i=1}^N x_i \cdot \sum_{j=1}^N a_{ij} \cdot \frac{\partial x_j}{\partial x_\ell}$$

notar que $\frac{\partial x_i}{\partial x_\ell} = \delta_{i\ell}$ y análogamente: $\frac{\partial x_j}{\partial x_\ell} = \delta_{j\ell}$

$$\therefore \frac{\partial f}{\partial x_\ell} = \sum_{i=1}^{N} \frac{\partial x_i}{\partial x_\ell} \cdot \sum_{j=1}^{N} a_{ij} \cdot x_j + \sum_{i=1}^{N} x_i \cdot \sum_{j=1}^{N} a_{ij} \frac{\partial x_j}{\partial x_\ell}$$

$$= \sum_{i=1}^{N} \delta_{i\ell} \cdot \sum_{j=1}^{N} a_{ij} \cdot x_j + \sum_{i=1}^{N} x_i \cdot \sum_{j=1}^{N} a_{ij} \cdot \delta_{j\ell}$$

$$\underset{\substack{\uparrow \\ \text{solo si} \\ i=\ell \text{ sumo} \neq 0}}{=} 1 \cdot \sum_{j=1}^{N} a_{\ell j} \cdot x_j + \sum_{i=1}^{N} x_i \cdot a_{i\ell} \cdot 1 = \sum_{j=1}^{N} a_{\ell j} \cdot x_j + \sum_{i=1}^{N} x_i \cdot a_{i\ell}$$

Como los índices son mudos, podemos juntar todo:

$$= \sum_{i=1}^{N} (a_{\ell i} + a_{i\ell})\, x_i \qquad \text{Como } A \text{ simétrica: } a_{\ell i} = a_{i\ell}$$

$$= 2 \sum_{i=1}^{N} a_{\ell i} \cdot x_i = 2 A_\ell \cdot x$$

$$\Rightarrow \nabla f = \begin{pmatrix} 2A_1 x \\ \vdots \\ 2A_N x \end{pmatrix} = 2 \begin{pmatrix} A_1 x \\ \vdots \\ A_N x \end{pmatrix} = 2Ax.$$

b) Si $g(x) = x^T x \underset{\substack{\uparrow \\ A=I}}{\Rightarrow} \nabla g(x) = 2 I x = 2x.$

P2) Tenemos $\nabla f = 0 \quad \forall x \in \Omega$. $\quad \gamma: [0,1] \to \Omega \subset \mathbb{R}^N$

Sea $\phi(t) = f(\gamma(t))$, $t \in [0,1]$; $\phi: [0,1] \to \mathbb{R}$. ie ϕ es fn. de 1 variable.

notar que $\frac{d}{dt}(\phi(t)) = \frac{d}{dt}(f(\gamma(t))) = \sum_{i=1}^{N} \frac{\partial f(\gamma(t))}{\partial x_i} \cdot \frac{\partial(\gamma(t))}{\partial t} = \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}(\gamma(t)) \cdot \gamma'(t)$

$$= \underbrace{\nabla f(\gamma(t)) \cdot \gamma'(t)}_{=0 \ \forall t \in [0,1]} = 0 \qquad \therefore \phi'(t) = 0 \quad \forall t \in [0,1]$$

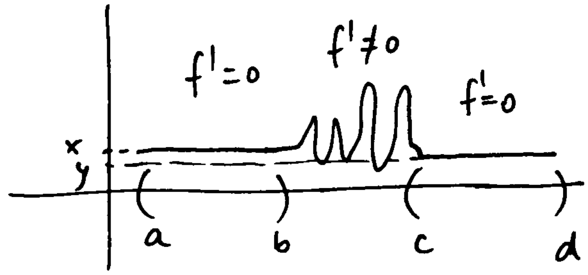
$\Rightarrow \phi \equiv \text{cte en } (0,1)$

$\Rightarrow \phi \equiv \text{cte en } [0,1]$ (cont.)

$\therefore \phi(0) = \phi(1) \Rightarrow f(x_0) = f(y) \ \forall y \Rightarrow f \equiv \text{cte}.$

Un contraejemplo: Sea $N=1 \Rightarrow$ conexo por caminos = intervalo.

Basta notar el sgte ejemplo



En $\Omega = (a,b) \cup (c,d)$

f' es 0 pero $f \neq$ cte.

en Ω $(x \neq y)$.

P3 $\Delta u = 0$.

Sup. $u(x) = v(r) = v\left(\sqrt{x_1^2 + \dots + x_N^2}\right)$

a) Pdq. $\Delta u(x) = v''(r) + \frac{N-1}{r} v'(r)$

Sol. Debemos ver que $\frac{\partial^2 u}{\partial x_1^2} + \dots + \frac{\partial^2 u}{\partial x_N^2} = v''(r) + \frac{N-1}{r} v'(r)$.

Calculemos $\frac{\partial^2 u}{\partial x_i^2}$: $\frac{\partial u}{\partial x_i} = \frac{\partial}{\partial x_i} v\left(\sqrt{x_1^2 + \dots + x_N^2}\right)$

$$= v'\left(\sqrt{x_1^2 + \dots + x_N^2}\right) \cdot \frac{\partial}{\partial x_i}\left(\sqrt{x_1^2 + \dots + x_N^2}\right)$$

$$= v'(r) \cdot \frac{1}{\cancel{2}} \cdot \frac{1}{r} \cdot \cancel{2} x_i = \frac{x_i}{r} v'(r)$$

$$\Rightarrow \frac{\partial^2 u}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(\frac{x_i}{r} v'(r)\right) = \frac{v'(r)}{r} + \frac{\partial}{\partial x_i}\left(\frac{v'(r)}{r}\right) \cdot x_i$$

$$= \frac{v'(r)}{r} + \frac{\partial}{\partial x_i}\left(\frac{1}{\sqrt{x_1^2 + \dots + x_N^2}} \cdot v'\left(\sqrt{x_1^2 + \dots + x_N^2}\right)\right) \cdot x_i$$

$$= \frac{v'(r)}{r} + \left(\frac{v''\left(\sqrt{x_1^2 + \dots + x_N^2}\right)}{\sqrt{x_1^2 + \dots + x_N^2}} \cdot \frac{x_i}{\sqrt{x_1^2 + \dots + x_N^2}} + \right.$$

$$\left. - \frac{1}{\cancel{2}} \cdot \frac{\cancel{2} x_i}{(\sqrt{\quad})^{3/2}} v'(\sqrt{\quad})\right) x_i$$

$$= \frac{v'(r)}{r} + \left(\frac{v''(r)}{r^2} \cdot x_i - \frac{v'(r) x_i^2}{r^3}\right) \Rightarrow \sum \frac{\partial^2 u}{\partial x_i^2} = \sum \frac{r^2 v'(r)}{r^3} - \frac{v'(r) x_i^2}{r^3} + \frac{v''(r) x_i^2}{r^2}$$

$$\Rightarrow \sum \frac{\partial^2 u}{\partial x_i^2} = \frac{N r^2 v'(r) - \overbrace{(x_1^2 + \dots + x_N^2)}^{r^2} v'(r)}{r^3} + \frac{v''(r)}{r^2} \underbrace{(x_1^2 + \dots + x_N^2)}_{r^2}$$

$$= \frac{(N-1)\cancel{r^2}\, v'(r)}{r^{\cancel{3}}} + \frac{v''(r)}{\cancel{r^2}} \cancel{r^2}$$

$$= \frac{(N-1)}{r} v'(r) + v''(r). \qquad \square$$

Para concluir lo otro hay que resolver la EDO

$$\frac{(N-1)}{r} v'(r) + v''(r) = 0$$

Si $N = 2$: EDO: $\frac{1}{r} v'(r) + v''(r) = 0$

Si $N \geq 2$: EDO $\frac{N-1}{r} v'(r) + v''(r) = 0$

el cálculo queda propuesto a ustedes :)

P4

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & (x,y) \neq (0,0). \\ 0 & (x,y) = (0,0). \end{cases}$$

a) f es dif en $\mathbb{R}^2$. (más aun C^1)

Calculemos las DP para $(x,y) \neq (0,0)$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\frac{xy(x^2-y^2)}{x^2+y^2}\right) = \frac{\partial}{\partial x}\left(\frac{x^3y - xy^2}{x^2+y^2}\right) = \frac{(x^2y - y^2)(x^2+y^2) - 2x(x^3y - xy^2)}{(x^2+y^2)^2}$$

$$= \frac{(x^4y + x^2y^3 - y^2x^2 - y^4 - 2x^4y + 2x^2y^2)}{(x^2+y^2)^2}$$

$$= \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2+y^2)^2}$$

análogamente: $\dfrac{\partial f}{\partial y} = \dfrac{x(x^4 - 4x^2y^2 - y^4)}{(x^2+y^2)^2}$

En $(0,0)$: $\dfrac{\partial f}{\partial x}(0,0) = \lim\limits_{t\to 0} \dfrac{f(t,0)}{t} = \lim\limits_{t\to 0} \dfrac{0}{t} = 0$ (análogo $\dfrac{\partial f}{\partial y}(0,0)$).

$$\therefore \frac{\partial f}{\partial x}(x,y) = \begin{cases} \dfrac{y(x^4 + 4x^2y^2 - y^4)}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$\frac{\partial f}{\partial y}(x,y) = \begin{cases} \dfrac{x(x^4 - 4x^2y^2 - y^4)}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

Si vemos que las DP son cont $\Rightarrow f$ es dif en $\mathbb{R}^2$ (y de clase C^1).

Para $\dfrac{\partial f}{\partial x}$: $\lim\limits_{(x,y)\to(0,0)} \dfrac{\partial f}{\partial x}(x,y) \overset{?}{=} \dfrac{\partial f}{\partial x}(0,0) = 0$ $\qquad 2(x^4 + 2x^2y^2 + y^2) = 2(x^2+y^2)^2$

Veamos que si:

$$0 \leq \left| \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2+y^2)^2} \right| \leq \frac{(x^2+y^2)(2x^4 + 4x^2y^2 + 2y^2)}{(x^2+y^2)^2} \leq \frac{2(x^2+y^2)\cancel{(x^2+y^2)^2}}{\cancel{(x^2+y^2)^2}}$$

$$= 2(x^2+y^2) \to 0$$

$\square$

$\therefore \frac{\partial f}{\partial x}$ es cont en $\mathbb{R}^2$.

Con $\frac{\partial f}{\partial y}$ es análogo

$\Rightarrow$ DP $\exists$ y son cont. en $\mathbb{R}^2 \Rightarrow f$ dif. en $\mathbb{R}^2$.

b) ~~f es dos veces dif en (0,0)~~. Calcule $\frac{\partial^2 f}{\partial y \partial x}(0,0)$ y $\frac{\partial^2 f}{\partial x \partial y}(0,0)$.

Es decir, hay que ver la $\exists$ de límites respectivos para $\frac{\partial f}{\partial x}$ y $\frac{\partial f}{\partial y}$

o sea: $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(0,0) = \lim_{t \to 0} \frac{\frac{\partial f}{\partial x}(t,0) - \frac{\partial f}{\partial x}(0,0)}{t} \overset{\text{expr de } \frac{\partial f}{\partial x}}{=} 0$

$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(0,0) = \lim_{t \to 0} \frac{\frac{\partial f}{\partial x}(0,t) - \frac{\partial f}{\partial x}(0,0)}{t} = \lim_{t \to 0} \frac{-t^5}{t^5} = -1$

$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(0,0) = \lim_{t \to 0} \frac{\frac{\partial f}{\partial y}(t,0) - \frac{\partial f}{\partial y}(0,0)}{t} = \lim_{t \to 0} \frac{t^5}{t^5} = 1$.

$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)(0,0) = \lim_{t \to 0} \frac{\frac{\partial f}{\partial y}(0,t) - \frac{\partial f}{\partial y}(0,0)}{t} = 0$.

c) ¿Qué puede concluir sobre f?

Se puede concluir que f no es C^2, i.e. que las DP de segundo orden no son continuas.