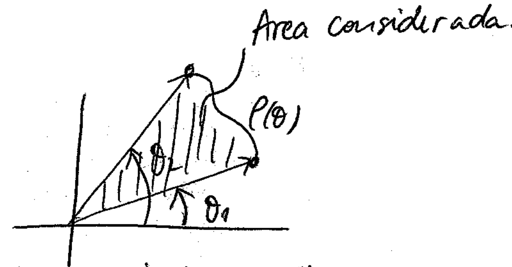
P1 Calculemos el área encerrada por las cardioides

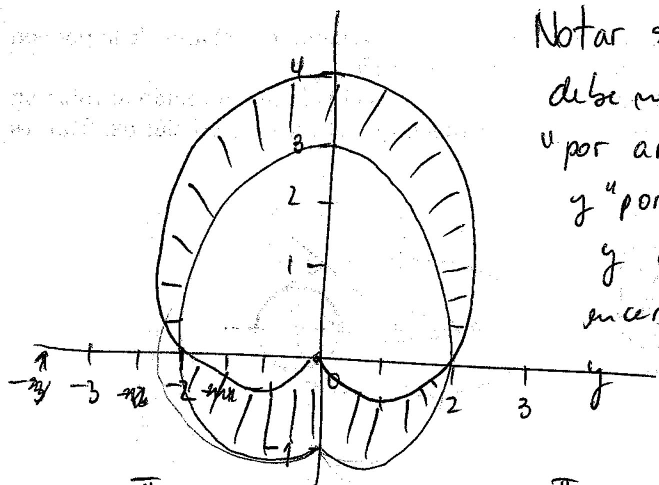
$$\rho = 2+\operatorname{sen}\theta \qquad \rho = 2+2\operatorname{sen}\theta.$$

Obs. Recordar que, en coord. polares:

$$\text{Área enc. entre } \theta=\theta_1 \text{ y } \theta=\theta_2 = \frac{1}{2}\int_{\theta_1}^{\theta_2}\rho^2(\theta)\,d\theta$$



Dibujemos las cardioides para entender lo que se desea integrar:



Notar que, entre $\theta=0$ y $\theta=\pi$ debemos calcular el área encerrada "por arriba" por $\rho = 2+2\operatorname{sen}\theta$ y "por abajo" por $\rho=2+\operatorname{sen}\theta$ y entre $\theta=\pi$ y $\theta=2\pi$ el área encerrada por "arriba" por $\rho=2+\operatorname{sen}\theta$ y "por abajo" por $\rho=2+2\operatorname{sen}\theta$.

$$\therefore A = \left(\frac{1}{2}\int_0^{\pi}(2+2\operatorname{sen}\theta)^2d\theta - \frac{1}{2}\int_0^{\pi}(2+\operatorname{sen}\theta)^2d\theta\right) + \left(\frac{1}{2}\int_{\pi}^{2\pi}(2+\operatorname{sen}\theta)^2d\theta - \frac{1}{2}\int_{\pi}^{2\pi}(2+2\operatorname{sen}\theta)^2d\theta\right)$$

$$= \frac{1}{2}\int_0^{\pi}(\cancel{4}+8\operatorname{sen}\theta+4\operatorname{sen}^2\theta-\cancel{4}-4\operatorname{sen}\theta-\operatorname{sen}^2\theta)\,d\theta$$

$$+\frac{1}{2}\int_{\pi}^{2\pi}(\cancel{4}+4\operatorname{sen}\theta+\operatorname{sen}^2\theta-\cancel{4}-8\operatorname{sen}\theta-4\operatorname{sen}^2\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi}4\operatorname{sen}\theta+3\operatorname{sen}^2\theta\,d\theta + \frac{1}{2}\int_{\pi}^{2\pi}(-4\operatorname{sen}\theta-3\operatorname{sen}^2\theta)\,d\theta$$

$$= 2(-\cos\theta)\Big|_0^{\pi} + \frac{3}{2}\int_0^{\pi}\frac{1-\cos 2\theta}{2}\,d\theta + 2\cos\theta\Big|_{\pi}^{2\pi} - \frac{3}{2}\int_{\pi}^{2\pi}\frac{1-\cos 2\theta}{2}\,d\theta$$

Suman 0.

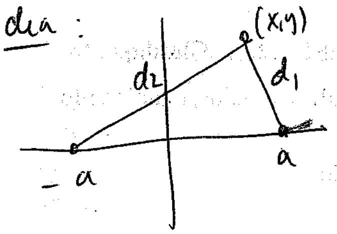
$$= 4+4 = 8$$

P2) $(x,y) \in L \iff d((x,y),(a,0)) \cdot d((x,y),(-a,0)) = a^2$.

Pdq $\rho^2(\theta) = \begin{cases} 2a^2\cos(2\theta) & \theta \in [0,\pi/4] \cup [3\pi/4, 5\pi/4] \cup [7\pi/4, \pi] \\ 0 & \end{cases}$ (revisar)

La idea es trabajar un poco algebraicamente la cond. geométrica y luego "pasarse" a coord. polares.

idea:



Si $(x,y) \in L$: $d_1 \cdot d_2 = a^2$.

Así pues, sup. que $(x,y) \in L$, luego:

$$d_1 \cdot d_2 = a^2 \iff d_1^2 \cdot d_2^2 = a^4$$

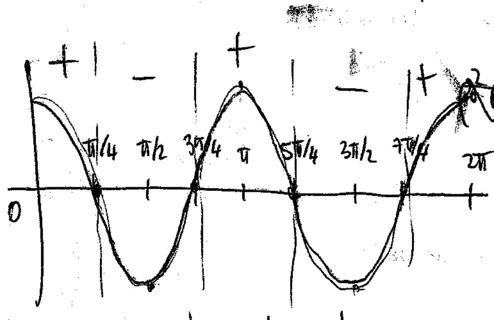
pero
$$d_1^2 = d^2((x,y),(a,0)) = \left(\sqrt{(x-a)^2+y^2}\right)^2 = (x-a)^2+y^2.$$
$$d_2^2 = d^2((x,y),(-a,0)) = \left(\sqrt{(x+a)^2+y^2}\right)^2 = (x+a)^2+y^2.$$

$$\therefore \left[(x-a)^2+y^2\right]\left[(x+a)^2+y^2\right] = a^4. \quad \text{si } (x,y) \in L.$$
$$= ((x-a)(x+a))^2 + y^2(x+a)^2 + y^2(x-a)^2 + y^4 = a^4.$$
$$= (x^2-a^2)^2 + y^2(2x^2+2a^2) + y^4 = a^4$$
$$= x^4 + \cancel{a^4} - 2x^2a^2 + 2x^2y^2 + 2a^2y^2 + y^4 = \cancel{a^4}$$
$$x^4 + 2x^2y^2 + y^4 = (y^2 - x^2)2a^2$$

Sean $x = \rho\cos\theta$, $y = \rho\,\mathrm{sen}\,\theta$.

$$\rho^4\cos^4\theta + \rho^4\,\mathrm{sen}^4\theta + 2\rho^4\cos^2\theta\,\mathrm{sen}^2\theta = \rho^2(\mathrm{sen}^2\theta - \cos^2\theta)2a^2$$

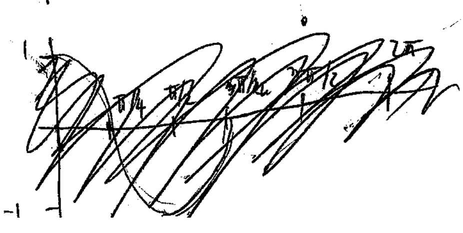
$$\cancel{\rho^2}\underbrace{\left(\rho^2\cos^4\theta + \rho^2\,\mathrm{sen}^4\theta + 2\rho^2\cos^2\theta\,\mathrm{sen}^2\theta\right)}_{\rho^2(\cos^2\theta+\mathrm{sen}^2\theta)^2} = \cancel{\rho^2}(\mathrm{sen}^2\theta - \cos^2\theta)2a^2$$
$$= 2a^2\cos(2\theta)$$



$$\rho^2 \cdot 1^2 = 2a^2\cos(2\theta)$$
$$\rho^2 = 2a^2\cos(2\theta)$$

Notar que esto vale solo donde $\cos(2\theta) \geq 0$

osea $\theta \in [0,\pi/4] \cup [3\pi/4, 5\pi/4] \cup [7\pi/4, 2\pi]$.



P3] a) $y = \varphi(t) \quad x = \psi(t)$

Param: $\vec{r}(t) = (\psi(t), \varphi(t)) \Rightarrow \frac{d\vec{r}}{dt} = (\psi'(t), \varphi'(t)) \quad t \in [a,b]$

$$\Rightarrow L(S) = \int_a^b \left\| \frac{dr}{dt} \right\| dt = \int_a^b \sqrt{\psi'^2(t) + \varphi'^2(t)}\, dt$$

En coord polares: $x(\theta) = \underbrace{\rho(\theta)\operatorname{sen}\theta}_{\psi(\theta)} \quad y(\theta) = \underbrace{\rho(\theta)\cos\theta}_{\varphi(\theta)}$

Con $\psi'(\theta) = \rho'(\theta)\operatorname{sen}\theta + \rho(\theta)\cos\theta \Rightarrow \psi'^2(\theta) = \rho'^2(\theta)\operatorname{sen}^2\theta + \rho^2(\theta)\cos^2\theta + 2\rho\rho'\operatorname{sen}\theta\cos\theta$

$\varphi'(\theta) = \rho'(\theta)\cos\theta - \rho(\theta)\operatorname{sen}\theta \Rightarrow \varphi'^2(\theta) = \rho'^2(\theta)\cos^2\theta + \rho^2(\theta)\operatorname{sen}^2\theta - 2\rho\rho'\operatorname{sen}\theta\cos\theta$

$$\therefore \psi'^2 + \varphi'^2 = \rho'^2(\operatorname{sen}^2\theta + \cos^2\theta) + \rho^2(\operatorname{sen}^2\theta + \cos^2\theta) + 0 = \rho'^2 + \rho^2$$

ie. $L(S) = \int_a^b \sqrt{\rho'^2(\theta) + \rho^2(\theta)}\, d\theta$

y si $y = f(x) \Rightarrow x(t) = t \wedge y(t) = f(x(t)) = f(t) \Rightarrow \vec{r}(t) = (t, f(t))$

$\Rightarrow \frac{d\vec{r}}{dt}(t) = (1, f'(t))$

$\Rightarrow L(S) = \int_a^b \sqrt{1 + f'^2(x)}\, dx$, que es obviamente la fórmula usual.

b) $x(t) = e^{2t}\cos(t) \quad y(t) = e^{2t}\operatorname{sen}(t) \quad t \in [0, 2\pi]$. $S = \{(x(t), y(t)) / t \in [0, 2\pi]\}$

$L(S) = \int_0^{2\pi} \sqrt{x'^2(t) + y'^2(t)}\, dt$ pero $\cancel{x'(t) = 2e^{2t}\cos(t) - e^{2t}\operatorname{sen}(t)}$

$\cancel{y'(t) =}$

notar que: $\left.\begin{matrix} x(t) = \rho(t)\cos(t) \\ y(t) = \rho(t)\sin(t) \end{matrix}\right\}$ polares!!

$\Rightarrow L(S) = \int_0^{2\pi} \sqrt{\rho'^2(t) + \rho^2(t)}\, dt$. $\quad \rho'(t) = 2e^{2t}$

$\Rightarrow \rho'^2(t) = 4e^{4t}, \quad \rho^2(t) = e^{4t}$

$$= \int_0^{2\pi} \sqrt{5e^{4t}}\, dt = \sqrt{5}\int_0^{2\pi} e^{2t}\, dt = \sqrt{5}\cdot\frac{1}{2}e^{2t}\Big|_0^{2\pi} = \frac{\sqrt{5}}{2}(e^{4\pi} - 1)$$

b) Queremos t_0 tq:

$$\int_0^{t_0} \sqrt{5e^{2t}}\, dt = \frac{\sqrt{5}}{4}(e^{4\pi} - 1)$$

$$\shortparallel$$

$$\frac{\sqrt{5}}{2}e^{2t_0} - \frac{\sqrt{5}}{2} = \frac{\sqrt{5}}{4}(e^{4\pi} - 1)$$

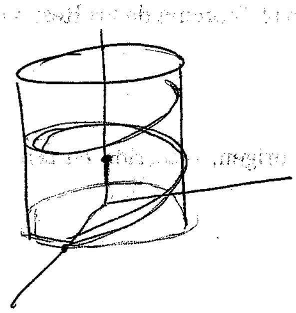
$$\frac{\sqrt{5}}{2}e^{2t_0} = \frac{\sqrt{5}}{4}e^{4\pi} + \frac{\sqrt{5}}{4}$$

$$e^{2t_0} = \frac{e^{4\pi} + 1}{2} \Rightarrow 2t_0 = \ln\left(\frac{e^{4\pi}+1}{2}\right)$$

$$\boxed{t_0 = \ln\left(\sqrt{\frac{e^{4\pi}+1}{2}}\right)}$$

Es lo buscado.

P4 | Manto del cil $x^2+y^2=1$ tq $z(\theta)$ satisf. $\frac{d^2z}{d\theta^2} = z$, $z(0)=1$, $\frac{dz}{d\theta}(0)=0$



a) Veamos (verifiquemos) que $z(\theta) = \cosh(\theta)$ satisface lo requerido:

$$\frac{d^2z}{d\theta^2} = \frac{d}{d\theta}\left(\frac{d}{d\theta}(\cosh(\theta))\right) = \frac{d}{d\theta}(\sinh(\theta)) = \cosh(\theta) = z(\theta) \checkmark$$

$$z(0) = \cosh(0) = 1 \checkmark, \quad \frac{dz}{d\theta}(0) = \frac{d(\cosh(\theta))}{d\theta}(0) = \sinh(0) = 0 \checkmark$$

b) Calculemos $L(S)$.

Notar que necesitamos una parametrización! Lo natural (dado que se mueve en un cilindro) es usar coord. cil: $x(\theta) = \rho(\theta)\cos\theta$, $y(\theta) = \rho(\theta)\operatorname{sen}\theta$, $z(\theta) = \cosh(\theta)$

pero! $x^2+y^2 = 1 \Rightarrow \rho = 1$ cte.

$\therefore r(\theta) = (\cos\theta, \operatorname{sen}\theta, \cosh\theta) \Rightarrow \frac{dr}{d\theta} = (-\operatorname{sen}\theta, \cos\theta, \sinh\theta)$ $\theta \in [0, 2\pi]$.

$$\Rightarrow L(S) = \int_0^{2\pi} \left\|\frac{dr}{d\theta}\right\| d\theta = \int_0^{2\pi} \sqrt{\operatorname{sen}^2\theta + \cos^2\theta + \sinh^2\theta} = \int_0^{2\pi} \sqrt{1+\sinh^2\theta} = \int_0^{2\pi} \cosh\theta\, d\theta$$

$$= \sinh(\theta)\Big|_0^{2\pi} = \sinh(2\pi)$$

P5 $a^2+b^2=c^2$

$$\vec{r}(s)=\left(a\cos\left(\frac{s}{c}\right), a\sin\left(\frac{s}{c}\right), b\left(\frac{s}{c}\right)\right) \quad \text{Pdq } s=L(s)$$

$s\in[0,2\pi c]$

$$\frac{d\vec{r}}{ds}=\left(-\frac{a}{c}\sin\left(\frac{s}{c}\right), \frac{a}{c}\cos\left(\frac{s}{c}\right), \frac{b}{c}\right) \Rightarrow \left\|\frac{d\vec{r}}{ds}\right\|=\left(\frac{a^2}{c^2}\left(\sin^2\left(\frac{s}{c}\right)+\cos^2\left(\frac{s}{c}\right)\right)+\frac{b^2}{c^2}\right)^{1/2}$$

$$=\left(\frac{a^2+b^2}{c^2}\right)^{1/2}=(1)^{1/2}\ \forall s.$$

$$=1\ \forall s$$

Como $\left\|\frac{d\vec{r}}{ds}\right\|=1\ \forall s$, se concluye

que la param es en longitud de arco.

P6 $x^2+y^2=z^2$.

$z(\theta)=e^{-\theta}$

Hay que usar cilíndricas!

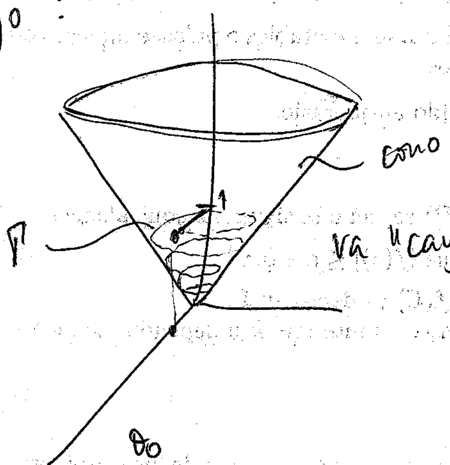
$\Rightarrow x(\theta)=\rho(\theta)\cos\theta \quad y(\theta)=\rho(\theta)\sin\theta \quad z(\theta)=e^{-\theta}$

pero $x^2+y^2=\rho^2(\theta)=z^2=e^{-\theta}$

$\Rightarrow \rho(\theta)=e^{-\theta}$

Param: $\vec{r}(\theta)=(e^{-\theta}\cos\theta,\ e^{-\theta}\sin\theta,\ e^{-\theta})$

a) Dibujo :



va "cayendo" hacia el origen.

b) $L(\Gamma)=\int_0^{\theta_0}\left\|\frac{dr}{d\theta}\right\|d\theta$ con $\frac{d\vec{r}}{d\theta}=\left(\frac{d}{d\theta}(\rho(\theta)\cos\theta),\ \frac{d}{d\theta}(\rho(\theta)\sin\theta),\ -e^{-\theta}\right)$

recordemos (P3) que $\left(\frac{d}{d\theta}(\rho(\theta)\cos\theta)\right)^2+\left(\frac{d}{d\theta}(\rho(\theta)\sin\theta)\right)^2=\rho'^2+\rho^2$

$$=\int_0^{\theta_0}\sqrt{\rho'^2+\rho^2+e^{-2\theta}}\,d\theta=\int_0^{\theta_0}\sqrt{e^{-2\theta}+e^{-2\theta}+e^{-2\theta}}\,d\theta$$

$$=\sqrt{3}\int_0^{\theta_0}e^{-\theta}d\theta=\sqrt{3}\cdot -e^{-\theta}\Big|_0^{\theta_0}$$

$$=\sqrt{3}\left(1-e^{-\theta_0}\right)$$

Notar que $L(\Gamma)\Big|_0^{\theta_0}=\sqrt{3}\left(1-e^{-\theta_0}\right)$

$\Rightarrow \lim_{\theta_0\to\infty}L(\Gamma)\Big|_0^{\theta_0}=\lim_{\theta_0\to\infty}\sqrt{3}\left(1-e^{-\theta_0}\right)=\sqrt{3}$, osea tiene largo finito!!

$\lim_{\theta_0\to\infty}e^{-\theta_0}=0$ (ya veremos este tipo de integrales más adelante) □