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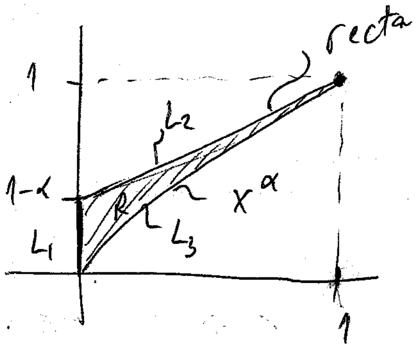
Aux: M. Godoy.

P1 (a) Recordemos el dibujo:



$\text{recta}(x) = m(x - x_0) + y_0$

Pasa por $(1,1)$, $m = (x^\alpha)'\big|_{x=1}$

$= \alpha(x^{\alpha-1})\big|_{x=1} = \alpha$

$\Rightarrow \text{recta}(x) = \alpha(x-1) + 1$

Caso $\alpha = \frac{2}{3} \Rightarrow \text{recta}(x) = \frac{2}{3}(x-1) + 1 = \frac{2}{3}x - \frac{2}{3} + 1 = \frac{2}{3}x + \frac{1}{3}$.

Queremos el perímetro de R

$$\Rightarrow P(R) = L_1 + L_2 + L_3 = (1-\overset{2/3}{\alpha}) + L_2 + L_3 = \frac{1}{3} + L_2 + L_3$$

L_2 = largo de recta(x) entre $x=0$ y $x=1$

L_3 = largo de $x^{2/3}$ entre $x=0$ y $x=1$.

$\Rightarrow$ Rcdo: Largo (longit de arco) de $f(x)$ entre $x=a$ y $x=b$

$$\| \quad \int_a^b \sqrt{1+f'(x)^2}\,dx$$

Así: $L_2 = \int_0^1 \sqrt{1+\text{recta}'(x)^2}\,dx$ pero $\text{recta}'(x) = \frac{2}{3}$ $\forall x$.

$$= \int_0^1 \sqrt{1+\left(\frac{2}{3}\right)^2}\,dx = \int_0^1 \sqrt{1+\frac{4}{9}}\,dx = \sqrt{\frac{13}{9}}$$

$$L_3 = \int_0^1 \sqrt{1+\left[\left(x^{\frac{2}{3}}\right)'\right]^2}\,dx = \int_0^1 \sqrt{1+\left(\frac{2}{3}\cdot x^{-1/3}\right)^2} = \int_0^1 \sqrt{1+\frac{4}{9}x^{-2/3}}\,dx$$

$$\Rightarrow L_3 = \int_0^1 \sqrt{1+\frac{4}{9}x^{-2/3}}\,dx = \int_0^1 \sqrt{x^{-2/3}\left(x^{2/3}+\frac{4}{9}\right)}\,dx = \int_0^1 \frac{1}{x^{1/3}}\sqrt{x^{2/3}+\frac{4}{9}}\,dx$$

Sea $u = x^{2/3}+\frac{4}{9} \Rightarrow du = \frac{2}{3}x^{-1/3}\,dx$

$$= \frac{3}{2}\int \sqrt{u}\,du = \frac{3}{2}\cdot\frac{1}{\frac{3}{2}}u^{3/2}$$

$$= \left(x^{2/3}+\frac{4}{9}\right)^{3/2}\Bigg|_0^1$$

$$= \left(\frac{13}{9}\right)^{3/2} - \left(\frac{4}{9}\right)^{3/2}$$

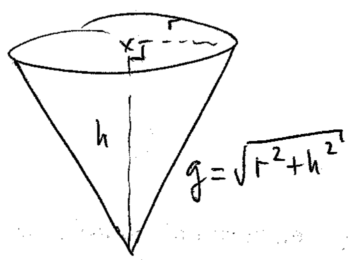
$$= \frac{\sqrt{13^3}}{27} - \frac{8}{27} = \frac{13\sqrt{13}-8}{27}.$$

Finalmente

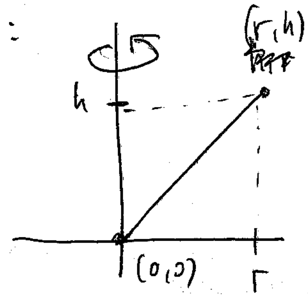
$$P(R) = \frac{1}{3} + \frac{\sqrt{13}}{3} + \frac{13\sqrt{13}-8}{27}$$

$$= \frac{9+9\sqrt{13}+13\sqrt{13}-8}{27} = \frac{1+22\sqrt{13}}{27}.$$

b) Pdq $S_{cono} = \pi r\sqrt{r^2+h^2} + \pi r^2$



notar que:



Al rotar en Oy obtengo el cono.

Ecc. recta: $y = mx$, $m = \frac{h}{r} \Rightarrow y = \frac{h}{r}x \Rightarrow f(x) = \frac{h}{r}x$ con $x \in [0,r]$

$\Rightarrow f' = \frac{h}{r}$

$$A_y(f)\Big|_0^r = 2\pi\int_0^r x\sqrt{1+f'^2(x)}\,dx = 2\pi\int_0^r x\sqrt{1+\frac{h^2}{r^2}}\,dx$$

$$= \cancel{2}\pi\sqrt{\frac{r^2+h^2}{r^2}}\cdot\frac{r^2}{\cancel{2}} = \pi r^{\cancel{2}}\frac{\sqrt{r^2+h^2}}{\cancel{r}}$$

$$A_y(f)\Big|_0^r = \pi r\sqrt{r^2+h^2}$$

Notar que esto no incluye la tapa, que es un círculo de radio r y por lo tanto su área es πr^2 $\Rightarrow S_{cono} = S_{manto} + S_{tapa}$

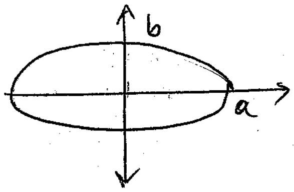
$$S_{cono} = \pi r\sqrt{r^2+h^2} + \pi r^2$$

como se deseaba.

P2

$$\mathcal{E}: \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$$\mathcal{E}_\lambda: \left(\frac{x}{\lambda a}\right)^2 + \left(\frac{y}{\lambda b}\right)^2 = 1$$



a) Para calcular s debemos despejar y en la ecc. de la elipse y aprovechar simetrías pues ~~[illegible]~~ $S = 4 \cdot$ Largo en 1er cuadrante. $\left(\text{o } f(x) = \frac{b}{a}\sqrt{a^2-x^2}\right)$

Así: $\left(\frac{y}{b}\right)^2 = 1 - \left(\frac{x}{a}\right)^2 \Rightarrow y = b\cdot\sqrt{1-\left(\frac{x}{a}\right)^2}$ (En el 1er cuadrante)

$x \in [0, a]$, $\quad = f(x)$

$$\therefore\ S = 4\cdot\int_0^a \sqrt{1+f'^2(x)}\,dx, \quad \text{calculemos } f'(x):\ f'(x) = b\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\cdot -2\left(\frac{x}{a}\right)\cdot\frac{1}{a}$$

$$\Rightarrow f'(x) = -\frac{b}{a}\frac{\left(\frac{x}{a}\right)}{\sqrt{1-\left(\frac{x}{a}\right)^2}} = -\frac{b}{a}\cdot\frac{x}{a}\cdot\frac{1}{\sqrt{\frac{a^2-x^2}{a^2}}} = -\frac{b}{a}\cdot\frac{x}{\cancel{a}}\cdot\frac{1}{\sqrt{a^2-x^2}}\cdot\cancel{a} = -\frac{b}{a}\frac{x}{\sqrt{a^2-x^2}}$$

Luego: $S = 4\displaystyle\int_0^a \sqrt{1+\left(\frac{b}{a}\right)^2\frac{x^2}{a^2-x^2}}\,dx$

$$= 4\int_0^a \sqrt{\frac{a^2-x^2+\left(\frac{b}{a}\right)^2x^2}{a^2-x^2}}\,dx$$

$\boxed{\leftarrow \text{mejor no, pues no podré cancelar fácilmente los } \lambda}$

$$= 4\int_0^a \sqrt{1+\left(\frac{b}{a}\right)^2\frac{x^2}{a^2-x^2}}\,dx = 4\int_0^a \sqrt{1+\left(\frac{b}{a}\right)^2\frac{1}{\left(\frac{a}{x}\right)^2-1}}\,dx$$

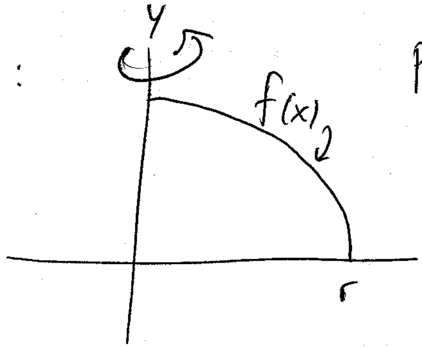
Análogamente: $S_\lambda = 4\displaystyle\int_0^{\lambda a} \sqrt{1+\left(\frac{\lambda b}{\lambda a}\right)^2\frac{1}{\left(\frac{\lambda a}{x}\right)^2-1}}\,dx$ $\quad$ Sea $u = x/\lambda \Rightarrow du = \frac{dx}{\lambda} \Rightarrow \lambda du = dx$

$$= 4\int_0^a \sqrt{1+\left(\frac{b}{a}\right)^2\frac{1}{\left(\frac{a}{u}\right)^2-1}}\,\lambda du$$

$$= 4\lambda\int_0^a \sqrt{1+\left(\frac{b}{a}\right)^2\cdot\frac{1}{\left(\frac{a}{u}\right)^2-1}}\,du = \lambda\cdot S \quad ✓$$

P3 a) $A_{esf} = 4\pi r^2$.

Notar que:



Por simetría

$$A_{esf} = 2\cdot A_y(f)$$

f? basta despejar: $x^2+y^2=r^2$

$$y=\sqrt{r^2-x^2} \quad (\text{Posit!})$$

$$f(x)=\sqrt{r^2-x^2}.$$

$$f'(x)=\frac{1}{2}\frac{(-2x)}{\sqrt{r^2-x^2}}=\frac{-x}{\sqrt{r^2-x^2}}.$$

$$\Rightarrow (f'(x))^2=\frac{x^2}{r^2-x^2}.$$

Luego $A_{esf} = \frac{4}{2}\int_0^r 2\pi x\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx = \frac{4\cdot 2\pi}{2}\int_0^r x\sqrt{\frac{r^2-\cancel{x^2}+\cancel{x^2}}{r^2-x^2}}\,dx$

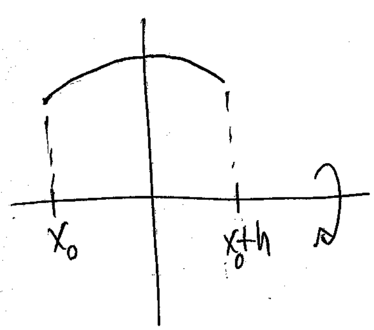
$$=\frac{8\pi}{2}\int_0^r x\cdot\frac{r}{\sqrt{r^2-x^2}}\,dx = \pi\frac{8}{2}r\int_0^r\frac{x}{\sqrt{r^2-x^2}}\,dx \qquad \text{si } u=r^2-x^2,\ du=-2x\,dx$$

$$=\frac{8\pi r}{2}\int_{r^2}^{0}\left(-\frac{1}{2}\right)\frac{du}{\sqrt{u}} = +\frac{4}{2}\pi r\int_0^{r^2}\frac{1}{\sqrt{u}}\,du = \frac{4\pi r}{\cancel{2}}\cdot\frac{1}{\cancel{\frac{1}{2}}}u^{1/2}\Big|_0^{r^2}$$

$$=4\pi r\cdot(r^2)^{1/2}$$

$$=4\pi r^2.$$

Veamos el "caso general"



Por simetría puedo rotar en OX y me dará el mismo resultado.

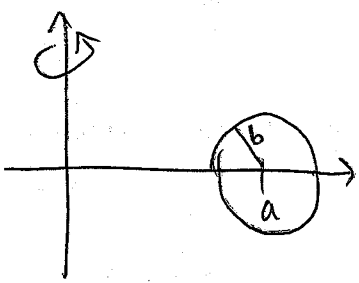
Sup pedida $= A_y(f)\Big|_{x_0}^{x_0+h}$ (Notación)

En tal caso: $A_y(f) = 2\pi\int_{x_0}^{x_0+h} f(x)\sqrt{1+f'^2(x)}\,dx$

$$\Rightarrow A_y(f)\Big|_{x_0}^{x_0+h} = 2\pi\int_{x_0}^{x_0+h} \cancel{\sqrt{r^2-x^2}}\cdot\frac{r}{\cancel{\sqrt{r^2-x^2}}}\,dx = 2\pi r\cdot(x_0+h-x_0)$$
$$= 2\pi r h \text{ !!}$$

obs. La primera Sup. calculada tb. podía hacerse de esta manera (check!)

b)



Ecc. circunferencia: $(x-a)^2+y^2=b^2$
(vale solo en $x\in[a-b,a+b]$)

despejemos y para obtener $f(x)$, $x\in[a-b,a+b]$ para integrar. Notar que por simetría tendremos

$$\cancel{A}\ S_{TORO} = 2A_y(f)\Big|_{a-b}^{a+b}$$

Así pues: $\boxed{y=\sqrt{b^2-(x-a)^2} = f(x)}$ Notar que: $f'(x) = \frac{1}{\cancel{2}\sqrt{b^2-(x-a)^2}}\cdot\left(-\cancel{2}(x-a)\right)$

$$f'(x) = \frac{-(x-a)}{\sqrt{b^2-(x-a)^2}}$$

Luego: $1+f'^2(x) = 1+\frac{(x-a)^2}{b^2-(x-a)^2}$

$$= \frac{b^2-\cancel{(x-a)^2}+\cancel{(x-a)^2}}{b^2-(x-a)^2}$$

$$\Rightarrow A_y(f)\Big|_{a-b}^{a+b} = 2\pi\int_{a-b}^{a+b} x\sqrt{1+f'^2(x)}\,dx = 2\pi\int_{a-b}^{a+b} x\cdot\sqrt{\frac{b^2}{b^2-(x-a)^2}}\,dx$$

$$= 2\pi\int_{a-b}^{a+b}\frac{bx}{\sqrt{b^2-(x-a)^2}}\,dx = 2\pi b\int_{a-b}^{a+b}\frac{x}{\sqrt{b^2-(x-a)^2}}\,dx$$

Sea $u = x-a$ $\Rightarrow du = dx$

$$= 2\pi b\int_{-b}^{b}\frac{u+a}{\sqrt{b^2-u^2}}\,du = 2\pi b\left[\int_{-b}^{b}\frac{u}{\sqrt{b^2-u^2}}\,du + a\int_{-b}^{b}\frac{du}{\sqrt{b^2-u^2}}\right]$$

$v=\arcsin\left(\frac{u}{b}\right)$
$\Uparrow$
$u = b\,\text{sen}\,v \Rightarrow du = b\cos v\,dv$

Sea y $\sqrt{b^2-u^2} = b\cos v$ $\Bigg\}\Rightarrow$ $\int\frac{u}{\sqrt{b^2-u^2}}\,du = \int\frac{b\,\text{sen}\,v}{\cancel{b\cos v}}\,\cancel{b\cos v}\,dv = b\int\text{sen}\,v = -b\cos v$
$= -b\cos\left(\arcsin\left(\frac{u}{b}\right)\right) = \boxed{-\sqrt{b^2-x^2}}$

$$\int\frac{du}{\sqrt{b^2-u^2}} = \int b\,dv = bv = \boxed{b\arcsin\left(\frac{u}{b}\right)}$$

$$\therefore \; A_y(f)\Big|_{a-b}^{a+b} = 2\pi b \left[a \cdot \arcsin\left(\frac{x}{b}\right)\Big|_{-b}^{b} \quad \sqrt{b^2 - x^2}\,\Big|_{-b}^{b} \right]$$

$$= 2\pi b \left[a\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) - 0 \right] = 2\pi^2 ab.$$

$$\therefore \; S_{TORO} = 2 \cdot 2\pi^2 ab = 4\pi^2 ab. \checkmark$$

P4) Cardioides: $\rho = 2 + \operatorname{sen}\theta \;\wedge\; \rho = 2 + 2\operatorname{sen}\theta$.

Se hará en la Auxiliar 11.