Pauta Auxiliar 6 - MA1002 - Calc. Dif. e Int.
2012/2 - Sección 5

P1 a) $\int e^{-x}\ln(1+e^x)\,dx$ La idea para esta primitiva es usar IPP.

¿Por qué? porque si $u = \ln(1+e^x) \Rightarrow u' = \dfrac{e^x}{1+e^x}$

$v' = e^{-x} \Rightarrow \underbrace{v = -e^{-x}}$

nos quedará una primitiva solo en términos de e^x!

$$\overset{IPP}{=} -e^{-x}\ln(1+e^x) - \int -\cancel{e^{-x}}\cdot\frac{\cancel{e^x}}{1+e^x}\,dx$$

$$= -e^{-x}\ln(1+e^x) + \underbrace{\int \frac{1}{1+e^x}\,dx}_{I}$$

Para calcular I usemos la sustitución $1+e^x = u \Rightarrow \underset{u-1}{e^x}dx = du \Rightarrow dx = \dfrac{du}{u-1}$

$$I = \int \frac{1}{1+e^x}dx = \int \frac{1}{u}\cdot\frac{du}{u-1} = \int\left(\frac{1}{u-1}-\frac{1}{u}\right)du = \int\frac{1}{u-1}du - \int\frac{1}{u}du$$

$$= \ln|u-1| - \ln|u| + C$$

$$= \ln|1+e^x-1| - \ln|1+e^x| + C$$

$$\underset{e^x,\ 1+e^x>0}{=} \ln e^x - \ln(1+e^x) + C$$

$$= x - \ln(1+e^x) + C$$

$\therefore$

$$\int e^{-x}\ln(1+e^x)\,dx = -e^{-x}\ln(1+e^x) + x - \ln(1+e^x) + C$$

b) $\displaystyle\int \frac{4x^3-3x^2+3}{(x-1)^2(x^2+1)}dx = \int \frac{p(x)}{q(x)}dx$ con $\operatorname{gr} p = 3$, $\operatorname{gr} q = 4$ $\Rightarrow \operatorname{gr} q > \operatorname{gr} p$ ✓

podemos usar Frac. parciales

Así pues, queremos obtener la sgte. descomposición:

$$\frac{4x^3-3x^2+3}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$$

ojo! Forma irreducible!

$$= \frac{A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2}{(x-1)^2(x^2+1)}$$

$$= \frac{A(x^3-x^2+x-1) + B(x^2+1) + (Cx+D)(x^2-2x+1)}{(x-1)^2(x^2+1)}$$

ordenando: $= \dfrac{x^3(A+C) + x^2(B-A-2C+D) + x(A+C-2D) + (B-A+D)}{(x-1)^2(x^2+1)}.$

$= \dfrac{4x^3 - 3x^2 + 3}{(x-1)^2(x^2+1)}$

$\therefore \quad A + C = 4$ (1), $B - A - 2C + D = -3$ (2), $A + C - 2D = 0$ (3), $B - A + D = 3$ (4)

(1) en (3): $4 - 2D = 0 \Rightarrow \boxed{D = 2}$ en (4): $B - A = 1.$ (5)

(5) en (2): $1 - 2C + 2 = -3 \Rightarrow 2C = 6 \Rightarrow \boxed{C = 3}$ en (1) $\Rightarrow \boxed{A = 1}$

en (4): $B - 1 + 2 = 3 \Rightarrow \boxed{B = 2}$

$$\therefore \quad \frac{4x^3 - 3x^2 + 3}{(x-1)^2(x^2+1)} = \frac{1}{(x-1)} + \frac{2}{(x-1)^2} + \frac{3x+2}{(x^2+1)}$$

$$\therefore \int \frac{4x^3 - 3x^2 + 3}{(x-1)^2(x^2+1)}\,dx = \int \frac{1}{x-1}\,dx + 2\int \frac{1}{(x-1)^2}\,dx + \int \frac{3x+2}{x^2+1}\,dx$$

pero: $\displaystyle\int \frac{dx}{x-1} = \ln|x-1|$, $\displaystyle\int \frac{1}{(x-1)^2}\,dx \underset{\substack{\uparrow \\ u = x-1 \\ du = dx}}{=} \int \frac{1}{u^2}\,du = \frac{-1}{u} = \frac{-1}{x-1}$

$$\int \frac{3x+2}{x^2+1}\,dx = \frac{3}{2}\int \frac{2x}{x^2+1} + 2\int \frac{1}{x^2+1}\,dx = \frac{3}{2}\ln(x^2+1) + 2\arctg(x)$$

$\leftarrow f'$, $\nwarrow f$

$$\therefore \int \frac{4x^3 - 3x^2 + 3}{(x-1)^2(x^2+1)}\,dx = \ln|x-1| + 2 \cdot \frac{(-1)}{x-1} + \frac{3}{2}\ln(x^2+1) + 2\arctg(x) + C.$$

c) $\displaystyle\int \frac{x}{\sqrt{x^2+1+(x^2+1)^{3/2}}}\,dx$ Sustit: $u = x^2+1 \Rightarrow du = 2x\,dx$

$$= \frac{1}{2}\int \frac{2x\,dx}{\sqrt{x^2+1+(x^2+1)^{3/2}}} = \frac{1}{2}\int \frac{du}{\sqrt{u + u^{3/2}}} = \frac{1}{2}\int \frac{du}{\sqrt{u}\sqrt{1+\sqrt{u}}}$$

Sea ahora $v = \sqrt{u} \Rightarrow dv = \dfrac{1}{2\sqrt{u}}\, du$

$$\Rightarrow \frac{1}{2}\int \frac{du}{\sqrt{u}\,(1+\sqrt{u})^{1/2}} = \int \frac{dv}{\sqrt{1+v}}$$ si finalmente $t = 1+v$

$$= \int \frac{dt}{\sqrt{t}} = \int t^{-1/2}\, dt \cancel{= \frac{2}{3} t^{3/2} + C = \frac{2}{3}(1+v)^{3/2} + C}$$

$$\cancel{= \frac{2}{3}(1+\sqrt{u})^{3/2} + C = \frac{2}{3}\left(1+\sqrt{1+x^2}\right)^{3/2} + C}$$

$$= 2\sqrt{t} + C = 2\sqrt{1+v} + C = 2\sqrt{1+\sqrt{u}} + C$$

$$= 2\sqrt{1+\sqrt{x^2+1}} + C.$$

P2 Probaremos que $\int \sec x\, dx = \ln|\sec x + \operatorname{tg} x| + C$

a) Probemos que: $\dfrac{1}{\cos x} = \dfrac{1}{2}\left[\dfrac{\cos x}{1+\sin x} + \dfrac{\cos x}{1-\sin x}\right]$

En efecto: $\dfrac{1}{2}\left[\dfrac{\cos x}{1+\sin x} + \dfrac{\cos x}{1-\sin x}\right] = \dfrac{1}{2}\left[\dfrac{\cos x(1-\sin x) + \cos x(1-\sin x)}{1-\sin^2 x}\right]$

$$= \frac{1}{2}\left[\frac{\cos x - \cancel{\cos x \sin x} + \cos x + \cancel{\cos x \sin x}}{\cos^2 x}\right] = \frac{1}{2}\left[\frac{2\cos x}{\cos^2 x}\right] = \frac{1}{\cos x} \checkmark$$

$$\therefore \int \sec x\, dx = \int \frac{1}{\cos x}\, dx = \frac{1}{2}\left(\int \frac{\cos x}{1+\sin x}\, dx + \int \frac{\cos x}{1-\sin x}\, dx\right)$$

pero, si $f = 1+\sin x \Rightarrow f' = \cos x$, si $g = 1-\sin x \Rightarrow g' = -\cos x$

$$= \frac{1}{2}\left(\int \frac{f'}{f}\, dx - \int \frac{-\cos x}{1-\sin x}\, dx\right) = \frac{1}{2}\left(\int \frac{f'}{f}\, dx - \int \frac{g'}{g}\, dx\right)$$

$$= \frac{1}{2}(\ln f - \ln g) + C$$

$$= \frac{1}{2}\left(\ln|1+\sin x| - \ln|1-\sin x|\right) + C$$

$$= \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C$$

$$= \frac{1}{2}\ln\frac{(1+\sin x)(1+\sin x)}{|1-\sin^2 x|} + C = \frac{2}{2}\ln\sqrt{\frac{(1+\sin x)^2}{\cos^2 x}} + C$$

$$= \ln\frac{|1+\sin x|}{|\cos x|} + C = \ln|\sec x + \operatorname{tg} x| + C \checkmark$$

b) Vía sustitución $t = tg\frac{x}{2}$.

Recordemos que bajo esta sustitución: $dt = \frac{1}{2}\sec^2\frac{x}{2}dx = \frac{1}{2}\left(tg^2\frac{x}{2}+1\right)dx$

además: $\operatorname{sen} x = \frac{2t}{1+t^2}$ $\quad \cos x = \frac{1-t^2}{1+t^2}$ $\qquad = \frac{1}{2}(t^2+1)dx \Rightarrow \frac{2}{t^2+1}dt = dx$

$\therefore$ En nuestro caso: $\int \frac{1}{\cos x}dx = \int \frac{1}{\frac{1-t^2}{\cancel{1+t^2}}}\frac{2}{\cancel{t^2+1}}dt = 2\int \frac{dt}{1-t^2} = 2\int \frac{1}{(1+t)(1-t)}dt$

$$\underset{\substack{\uparrow\\ \text{verificar!}}}{=} \cancel{2}\int \frac{1}{\cancel{2}}\left(\frac{1}{1-t}+\frac{1}{1+t}\right)dt = -\ln|1-t| + \ln|1+t| + C$$

$$= -\ln\left|1-tg\frac{x}{2}\right| + \ln\left|1+tg\frac{x}{2}\right| + C$$

$$= \ln\left|\frac{1+tg\frac{x}{2}}{1-tg\frac{x}{2}}\right| + C$$

Veamos finalmente que:

$\left|\frac{1+tg\frac{x}{2}}{1-tg\frac{x}{2}}\right| = |\sec x + tg x|$, en efecto: $\left|\frac{1+tg\frac{x}{2}}{1-tg\frac{x}{2}}\right| = \left|\frac{1+\frac{\operatorname{sen} x/2}{\cos x/2}}{1-\frac{\operatorname{sen} x/2}{\cos x/2}}\right|$

$$= \left|\frac{\frac{\cos x/2 + \operatorname{sen} x/2}{\cancel{\cos x/2}}}{\frac{\cos x/2 - \sin x/2}{\cancel{\cos x/2}}}\right| = \left|\frac{\cos x/2 + \operatorname{sen} x/2}{\cos x/2 - \sin x/2}\right| \cdot \left|\frac{\cos x/2 + \sin x/2}{\cos x/2 + \sin x/2}\right| = \frac{|\cos x/2 + \sin x/2|^2}{|\cos^2 x/2 - \sin^2 x/2|}$$

pero $\cos 2x = \cos^2 x - \sin^2 x$

$$= \frac{|\cos x/2 + \sin x/2|^2}{|\cos x|} = \frac{|\overbrace{\cos^2 x/2 + \sin^2 x/2}^{1} + \overbrace{2\sin x/2\cos x/2}^{\sin x}|}{|\cos x|} = \left|\frac{1+\sin x}{\cos x}\right| = |\sec x + tg x| \checkmark$$

P3) $I_n = \int e^{-x} f^{(n)}(x)dx$

a) Para las recurrencias <u>SIEMPRE</u> se integra por partes

$$I_n = \int \underbrace{e^{-x}}_{v'} \underbrace{f^{(n)}(x)}_{u} dx = -e^{-x}f^{(n)}(x) - \int -e^{-x}f^{(n+1)}(x) = -e^{-x}f^{(n)}(x) + \int e^{-x}f^{(n+1)}(x)dx$$

$$= I_{n+1} - e^{-x}f^{(n)}(x) \checkmark$$

(pues quiero $f^{(n+1)}(x)$)

$\Rightarrow v = -e^{-x} \quad u' = f^{(n+1)}(x)$

b) de a): $I_{n+1} - I_n = e^{-x}f^{(n)}(x)$, si $f^{(k)} \equiv 0 \Rightarrow f^{(l)} \equiv 0 \ \forall l \geq k$

$\Rightarrow I_l = c \ \forall l \geq k$

$\underset{l\geq k}{\Rightarrow} \underbrace{\sum_{n=0}^{l}(I_{n+1} - I_n)}_{\text{Telescópica}} = \sum_{n=0}^{l} e^{-x}f^{(n)}(x) = \sum_{n=0}^{k-1} e^{-x}f^{(n)}(x) + C_2$

$I_{l+1} - I_0 = C_1 - I_0$

$\Rightarrow \boxed{I_0 = -\sum_{n=0}^{k-1} e^{-x}f^{(n)}(x) + \tilde{C}}$

$(\tilde{C} = C_1 - C_2$, cte.$)$

P4) a) $I_n = \int \frac{x^n}{\sqrt{1+x}}\,dx$

<u>Pdq.</u> $(1+2n)I_n = 2x^n\sqrt{1+x} - 2n I_{n-1}$

Como siempre, hay que integrar por partes:

$$\int \underset{u}{\underset{\uparrow}{x^n}} \cdot \underset{v'}{\underset{\uparrow}{\frac{1}{\sqrt{1+x}}}}\,dx = 2x^n\sqrt{1+x} - \int n x^{n-1} \cdot 2\sqrt{1+x}\,dx$$

$$= 2x^n\sqrt{1+x} - 2n\underbrace{\int x^{n-1}\sqrt{1+x}\,dx}_{\text{esto "no se parece" a } I_j \text{, algún } j.}$$

$u = x^n \Rightarrow u' = n x^{n-1}$

$v' = \frac{1}{\sqrt{1+x}} \Rightarrow v = 2\sqrt{1+x}$

Notar que: $\int x^{n-1}\sqrt{1+x}\,dx = \int x^{n-1}\sqrt{1+x}\cdot\frac{\sqrt{1+x}}{\sqrt{1+x}}\,dx$

$$= \int x^{n-1}\frac{|1+x|}{\sqrt{1+x}}\,dx \overset{\text{sup. } x \geq -1.}{=} \int \frac{x^{n-1}}{\sqrt{1+x}}\,dx + \int \frac{x^n}{\sqrt{1+x}}\,dx$$

$$\therefore\ I_n = 2x^n\sqrt{1+x} - 2n\left(\underbrace{\int\frac{x^{n-1}}{\sqrt{1+x}}\,dx}_{I_{n-1}} + \underbrace{\int\frac{x^n}{\sqrt{1+x}}\,dx}_{I_n}\right)$$

$$I_n = 2x^n\sqrt{1+x} - 2n I_{n-1} - 2n I_n$$

$$\therefore\ (1+2n)I_n = 2x^n\sqrt{1+x} - 2n I_{n-1}$$

que era lo deseado. :)

b) $\int \frac{\sin x}{1+\sin x}\,dx$ notemos que en este caso estamos "forzados" a usar la "técnica infalible" (pero tediosa) de hacer la sustitución $t = \operatorname{tg}\frac{x}{2}$ pues la primitiva no es directa (si tuviesemos alguno de los $\operatorname{sen} x$ con $\cos x$ sería directa, ¿Por qué?)

Así pues, si $t = \operatorname{tg}\frac{x}{2} \Rightarrow \operatorname{sen} x = \frac{2t}{1+t^2}$ $\quad dx = \frac{2}{t^2+1}\,dt$, luego:

$$\int \frac{\sin x}{1+\sin x}\,dx = \int \frac{\frac{2t}{1+t^2}}{1+\frac{2t}{1+t^2}} \cdot \frac{2}{t^2+1}\,dt = \int \frac{\frac{2t}{1+t^2}\cdot\frac{2}{1+t^2}}{\frac{1+t^2+2t}{1+t^2}}\,dt$$

$$= \int \frac{4t\cancel{(1+t^2)}}{(1+t^2)^{\cancel{2}}\underbrace{(1+2t+t^2)}_{(1+t)^2}}\,dt = \int \frac{4t}{(1+t^2)(1+t)^2}\,dt$$

Para esta nueva primit. basta hacer Frac. parciales!

$$\frac{4t}{(1+t^2)(1+t)^2} = \frac{A}{1+t} + \frac{B}{(1+t)^2} + \frac{Ct+D}{t^2+1}$$

$$= \frac{t^3(A+C) + t^2(B+A+2C+D) + t(A+C+2D) + (B+A+D)}{(1+t)^2(1+t^2)}$$

mismo calculo P1 b)

$\Rightarrow A+C=0$ (1), $\quad B+A+2C+D=0$ (2), $\quad A+C+2D=4$ (3), $\quad B+A+D=0$ (4)

(1) en (3) $\Rightarrow +2D=4 \Rightarrow \boxed{D=+2}$, $\quad$ (4) en (2) $\Rightarrow +2C=0 \Rightarrow \boxed{C=0}$

en (1) $\Rightarrow \boxed{A=0}$

Todos los result. en (4): $B+A+D=0 \Rightarrow B+0+(+2)=0$

$\Rightarrow \boxed{B=-2}$

$$\therefore \frac{4t}{(1+t^2)(1+t)^2} = \frac{(-2)}{(1+t)^2} + \frac{(+2)}{t^2+1}$$

$$\Rightarrow \int \frac{4t}{(1+t^2)(1+t^2)}\,dt = -2\underbrace{\int\frac{dt}{(1+t)^2}}_{-\frac{1}{(1+t)}} + 2\underbrace{\int\frac{dt}{t^2+1}}_{\operatorname{arctg}(t)} = \frac{+2}{1+t} + 2\operatorname{arctg}(t) + C$$

$$= \frac{+2}{1+\operatorname{tg}\frac{x}{2}} + 2\cdot\frac{x}{2} + C = \frac{+2}{1+\operatorname{tg}\frac{x}{2}} + x + C$$