PAUTA AUX 4

P1

(a) Para ver que $C=1$ notamos que la densidad $f_X(x)$ debe cumplir que $\int_{-\infty}^{+\infty} f_X(x) = 1$

$$1=\int_{-\infty}^{+\infty} f_X(x) = \int_{-1}^{0} -x\,dx + \int_{0}^{1} (C-x)\,dx$$

$$= -\frac{x^2}{2}\Big|_{-1}^{0} + \left(Cx - \frac{x^2}{2}\right)\Big|_{0}^{1}$$

$$= \frac{1}{2} + C - \frac{1}{2} = C \implies C=1$$

(b) Queremos calcular $F_X(x)$ $\forall x \in \mathbb{R}$

Por la definición de $f_X(x)$ vemos que

$\forall x < -1 \quad F_X(x) = 0$

$\forall x \geq 1 \quad F_X(x) = 1$

Si $x \in (-1, 0)$

ANTES DE -1 VALE CERO

$$\mathbb{P}(X \leq x) = F_X(x) = \int_{-\infty}^{x} -t\,dt = \int_{-1}^{x} -t\,dt$$

$$= -\frac{t^2}{2}\Big|_{-1}^{x} = \frac{1}{2} - \frac{x^2}{2}$$

Si $x \in (0, 1]$

$$\mathbb{P}(X \leq x) = F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$$

$$= \int_{-1}^{0} f_X(t)\,dt + \int_{0}^{x} f_X(t)\,dt$$

$$= \int_{-1}^{0} -t\,dt + \int_{0}^{x} (c-t)\,dt$$

$$= -\frac{t^2}{2}\Big|_{-1}^{0} + \left(t - \frac{t^2}{2}\right)\Big|_{0}^{x} = \frac{1}{2} + x - \frac{x^2}{2}$$

$C = 1$

(c) Analicemos la distribución de Y.

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(|X| \leq y)$$

$$= \mathbb{P}(-y \leq X \leq y)$$

$|X| \leq y \Leftrightarrow -y \leq X \leq y$

Notar que si $y > 1$ $F_Y(y) = 1$

Si $y < 0$ no tiene sentido, $F_Y(y) = 0$

$$= \mathbb{P}(X \leq y) - \mathbb{P}(X \leq -y) = F_X(y) - F_X(-y)$$

$y \in [0, 1]$

$$= \frac{1}{2} + y - \frac{y^2}{2} - \left(\frac{1}{2} - \frac{y^2}{2}\right)$$

Por (b)

$$= y.$$

$\Rightarrow \forall y \in [0, 1]$ $\quad f_Y(y) = 1.$ $\quad$ (Recordar que $f_Y(y) = \frac{d}{dy}(F_Y(y))$)

P2 $X_1 \sim exp(\lambda_1)$
$X_2 \sim exp(\lambda_2)$ indep.

Veamos la distribución de Y

$$F_Y(y) = \mathbb{P}(\min\{X_1, X_2\} \leq y)$$

$$= 1 - \mathbb{P}(\min\{X_1, X_2\} \geq y)$$

$$= 1 - \mathbb{P}(X_1 \geq y,\ X_2 \geq y)$$

$$= 1 - \mathbb{P}(X_1 \geq y)\,\mathbb{P}(X_2 \geq y)$$

$$= 1 - (1 - \mathbb{P}(X_1 \leq y))(1 - \mathbb{P}(X_2 \leq y)) \quad (*)$$

Recordar que si $Z \sim exp(\mu)$

$$F_Z(t) = 1 - e^{-\mu t}$$

Por lo tanto en $(*)$

$$(*) = 1 - (e^{-\lambda_1 y})(e^{-\lambda_2 y})$$

$$= 1 - e^{-(\lambda_1 + \lambda_2) y}$$

$\therefore$ la densidad es

$$f_Y(y) = \frac{d}{dy} F_Y(y) = (\lambda_1 + \lambda_2)\, e^{-(\lambda_1 + \lambda_2) y}$$

$$\Rightarrow \min\{X_1, X_2\} \sim exp(\lambda_1 + \lambda_2)$$

P3

$X \sim \exp(\lambda)$

con $k \in \mathbb{N}$

(a) $\mathbb{P}(\lfloor X \rfloor = k) = \mathbb{P}(k \leq X < k+1)$

$$= \int_k^{k+1} f_X(x)\,dx = \int_k^{k+1} \lambda e^{-\lambda x}\,dx = \left(-e^{-\lambda x}\right)\Big|_k^{k+1}$$

$$= e^{-\lambda k} - e^{-\lambda(k+1)}$$

(b) Notemos que para una exponencial se tiene que

$$\mathbb{P}(X > y) = 1 - \mathbb{P}(X \leq y) = 1 - \int_{-\infty}^{y} \lambda e^{-\lambda t} \mathbb{1}_{(0,\infty)}\,dt = 1 - \int_0^y \lambda e^{-\lambda t}\,dt$$

$$= 1 - \left(-e^{-\lambda t}\right)\Big|_0^y = 1 - \left(1 - e^{-\lambda y}\right)$$

$$= e^{-\lambda y}$$

$\{X > t+s\} \subseteq \{X > s\}$

$$\therefore \mathbb{P}(X > t+s \mid X > s) = \frac{\mathbb{P}(X > t+s,\, X > s)}{\mathbb{P}(X > s)} = \frac{\mathbb{P}(X > t+s)}{\mathbb{P}(X > s)}$$

$$= \frac{e^{-(t+s)\lambda}}{e^{-s\lambda}} = e^{-t\lambda} = \mathbb{P}(X > t)$$

|P4|

$$\mathbb{E}\left(\frac{1}{1+X}\right) = \sum_{x \in R_X} g(x)\, \mathbb{P}(X = x)$$

$$= \sum_{k=0}^{m} g(k)\, \mathbb{P}(X=k) = \sum_{k=0}^{m} \frac{1}{1+k} \underbrace{\binom{m}{k} p^k (1-p)^{m-k}}_{\mathbb{P}(X=k)}$$

$$g(x) = \frac{1}{1+x} \qquad X \sim Bin(m,p)$$

$$= \sum_{k=0}^{m} \frac{m!}{(m-k)!\,k!} \frac{p^k (1-p)^{m-k}}{(k+1)}$$

$$= \frac{1}{m+1} \sum_{k=0}^{m} \frac{(m+1)!}{((m+1)-(k+1))!\,(k+1)!}\, p^k (1-p)^{m-k}$$

$$= \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k+1} p^k (1-p)^{m-k}$$

$$= \frac{1}{m+1} \sum_{k=1}^{m+1} \binom{m+1}{k} p^{k-1} (1-p)^{m-(k-1)}$$

$$= \frac{1}{m+1} \left(\sum_{k=0}^{m+1} \binom{m+1}{k} \frac{p^k}{p} (1-p)^{(m+1)-k} - \frac{(1-p)^{m+1}}{p} \right)$$

$$= \frac{1}{m+1} \left(\frac{1}{p} \left(\sum_{k=0}^{m+1} \binom{m+1}{k} p^k (1-p)^{(m+1)-k} - (1-p)^{m+1} \right) \right)$$

$$= \frac{1}{p(m+1)} \left((p+1-p)^{m+1} - (1-p)^{m+1} \right)$$

$$= \frac{1-(1-p)^{m+1}}{p\,(m+1)}$$

|P5| Si X es una v.a. discreta

$$\mathbb{E}(X) = \sum_{k \in R_X} k \, \mathbb{P}(X=k)$$

Entonces podemos tomar una variable que

$$R_X = \{1, 2, \ldots\} \qquad \mathbb{P}(X=k) = \frac{6}{\pi^2 k^2}$$

$\rightarrow$ Esto vale de que sabemos $\displaystyle\sum_{k=1}^{+\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$

$$\Longrightarrow \sum_{k=1}^{+\infty} \frac{6}{\pi^2 k^2} = 1$$

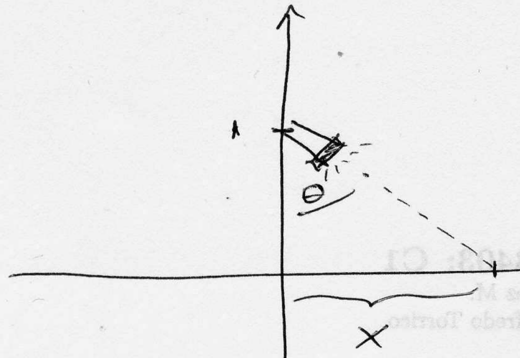
Dado lo anterior

$$\mathbb{E}(X) = \sum_{k \in R_X} k \, \mathbb{P}(X=k) = \sum_{k=1}^{+\infty} k \frac{6}{\pi^2 k^2}$$

$$= \frac{6}{\pi^2} \sum_{k=1}^{+\infty} \frac{1}{k}$$

Suma que sabemos que no converge.

P6



Notemos que $tg(\theta) = X$ y sabemos que

$$\theta \sim U\left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right)$$

Buscamos $F_X(t)$ y $f_X(t)$

$$F_X(t) = \mathbb{P}(X \le t) = \mathbb{P}(tg(\theta) \le t)$$

$$= \mathbb{P}(\theta \le arctg(t)) = \int_{-\infty}^{arctg(t)} f_\theta(u)\, du$$

tg(·) biyectiva en $(-\pi/2, \pi/2)$

$$= \int_{-\pi/2}^{arctg(t)} \frac{1}{\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)}\, du = \frac{1}{\pi}\left(arctg(t) - \left(-\tfrac{\pi}{2}\right)\right)$$

con $-\frac{\pi}{2} < arctg(t) < \frac{\pi}{2}$

$$= \frac{arctg(t)}{\pi} + \frac{1}{2} \quad \forall t \in (-\infty, +\infty)$$

$$\Rightarrow f_X(t) = \frac{d}{dt} F_X(t) = \frac{d}{dt}\left(\frac{arctg(t)}{\pi} + \frac{1}{2}\right) = \frac{1}{\pi(1+t^2)}$$

Veamos ahora

$$E(X) = \int_{-\infty}^{+\infty} t\, f_X(t)\, dt = \int_{-\infty}^{+\infty} \frac{t}{\pi(1+t^2)}\, dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{2t}{1+t^2}\, dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left(\ln(1+t^2)\right)' dt = \frac{1}{2\pi}\left(\ln(1+t^2)\Big|_{-\infty}^{+\infty}\right) \ (*)$$

$(*)$ ES INDEFINIDO PORQUE

$$(*) = \lim_{a\to+\infty} \frac{1}{2\pi}\left(\ln(1+t^2)\right)\Big|_{-2a}^{a}$$

$$= \lim_{a\to+\infty} \frac{1}{2\pi}\left(\ln(1+a^2) - \ln(1+4a^2)\right)$$

$$= \frac{1}{2\pi}\lim_{a\to+\infty} \ln\left(\frac{1+a^2}{1+4a^2}\right) = \frac{1}{2\pi}\ln\left(\frac{1}{4}\right)$$

y también

$$(*) = \lim_{a\to+\infty} \frac{1}{2\pi}\left(\ln(1+t^2)\right)\Big|_{-a}^{a}$$

$$= \lim_{a\to+\infty} \frac{1}{2\pi}\left(\ln(1+a^2) - \ln(1+a^2)\right) = 0.$$

[P7]. Definamos

$$X_i = \begin{cases} 1 & \text{la bolita } i \text{ es blanca} \\ 0 & \sim \end{cases}$$

$$\mathbb{P}(X_i = 1) = p = \frac{m}{N}$$

$$X = \sum_{i=1}^{n} X_i = \text{cantidad de bolitas blancas (al sacar } n\text{)}$$

$$\mathbb{E}(X) = \mathbb{E}\left(\sum_i X_i\right) \underset{\text{linealidad}}{=} \sum_{i=1}^{n} \mathbb{E}(X_i) = \sum_{i=1}^{n} \frac{m}{N} = \boxed{\frac{m}{N} \cdot n}$$

$$\mathbb{E}(X_i) = 1 \cdot \mathbb{P}(X_i = 1) + 0 \cdot \mathbb{P}(X_i = 0) = 1 \cdot p = \frac{m}{N}$$