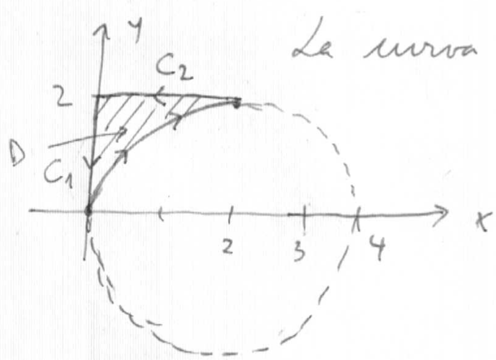
Pauta Ejercicio 1 - Cálculo Avanzado y Aplicaciones

Otoño 2012

1) La curva γ



Sea D la región achurada. Por teorema de Green

$$\int_{\partial D} \left(\frac{x-2}{(x-2)^2+(y-1)^2} - 2y \right) dx + \frac{y-1}{(x-2)^2+(y-1)^2} \, dy$$

$$= \iint_D \left(\frac{\partial}{\partial x} \frac{y-1}{(x-2)^2+(y-1)^2} - \frac{\partial}{\partial y} \left(\frac{x-2}{(x-2)^2+(y-1)^2} - 2y \right) \right) dA$$

$$= \iint_D \left(\frac{2(y-1)(x-2)}{((x-2)^2+(y-1)^2)^2} - \frac{2(x-2)(y-1)}{((x-2)^2+(y-1)^2)^2} + 2 \right) dA$$

$$= 2 \text{ área de } D = 2\left(4 - \tfrac{1}{4}\pi 2^2\right) = 2(4-\pi)$$

∂D se puede descomponer como γ y las 2 curvas de la figura, C_1 y C_2.

Calculamos

$$\int_{C_1} \left(\frac{x-2}{(x-2)^2+(y-1)^2} - 2y \right) dx + \frac{y-1}{(x-2)^2+(y-1)^2}\, dy$$

$$= - \int_0^2 \frac{y-1}{4+(y-1)^2}\, dy = - \frac{1}{2} \log\left(4+(y-1)^2\right) \Big|_0^2$$

$$= -\frac{1}{2}\left(\log 5 - \log 5\right) = 0.$$

$$\int_{C_2} \left(\frac{x-2}{(x-2)^2+(y-1)^2} - 2y \right) dx + \frac{y-1}{(x-2)^2+(y-1)^2}\, dy$$

$$= - \int_0^2 \left(\frac{x-2}{(x-2)^2+1} - 4 \right) dx = - \frac{1}{2} \log\left((x-2)^2+1\right) \Big|_0^2 + 8$$

$$= -\frac{1}{2}\left(\log(1) - \log(5)\right) + 8 = 8 + \frac{1}{2}\log 5$$

Entonces

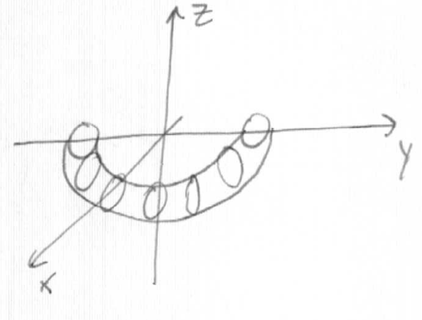
$$\int_{\gamma} \left(\frac{x-2}{(x-2)^2+(y-1)^2} - 2y \right) dx + \frac{y-1}{(x-2)^2+(y-1)^2}\, dy$$

$$= 2(4-\pi) - 8 - \frac{1}{2}\log 5$$

$$= -\frac{1}{2}\log 5 - 2\pi.$$

2) $\vec{F} = (z^4 + xz, zy, z)$

La superficie es parte de un toro



Definamos $D = \{(x,y,z) : (\sqrt{x^2+y^2} - R)^2 + z^2 < r, x > 0\}$

que es medio toro sólido.

Entonces $\partial D = \Sigma \cup T_1 \cup T_2$ donde

T_1 = disco de centro $(0, -R, 0)$ y radio r, en el plano yz

T_2 = disco de centro $(0, R, 0)$ y radio r, en el plano yz

Por teorema de Gauss

$$\iint_{\partial D} \vec{F} \cdot \hat{n} \, dA = \iiint_D \operatorname{div}(\vec{F}) \, dV$$

$$\operatorname{div}(\vec{F}) = z + z + 1 = 1 + 2z$$

$$\iiint_D \operatorname{div}(\vec{F}) \, dV = \operatorname{vol}(D) + 2 \iiint_D z \, dV$$

La última integral es cero porque D es simétrico con respecto al eje z. Así

$$\iiint_D \operatorname{div}(\vec{F}) = \operatorname{vol}(D) = \pi^2 R r^2.$$

Como $\partial D = \Sigma \cup T_1 \cup T_2$, calculamos

$$\iint_{T_1} \vec{F}\cdot\hat{n}\, dA \quad y \quad \iint_{T_2} \vec{F}\cdot\hat{n}\, dA$$

Observemos que T_1, T_2 deben orientarse según $-\hat{z}$

$$\iint_{T_1} \vec{F}\cdot\hat{n}\, dA = -\iint_{T_1} z^4\, dA$$

$$= -\int_0^r \int_0^{2\pi} (\rho \operatorname{sen}\theta)^4 \rho \, d\theta\, d\rho$$

$$= -\int_0^r \rho^5\, d\rho \int_0^{2\pi} \operatorname{sen}^4\theta\, d\theta$$

$$= -\frac{1}{6} r^6 \int_0^{2\pi} \operatorname{sen}^4\theta\, d\theta.$$

$$\cos 2\theta = \cos^2\theta - \operatorname{sen}^2\theta$$

$$= 1 - 2\operatorname{sen}^2\theta$$

$$\operatorname{sen}^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\int_0^{2\pi} \operatorname{sen}^4\theta\, d\theta = \int_0^{2\pi} \left(\frac{1-\cos 2\theta}{2}\right)^2 d\theta$$

$$= \frac{1}{4}\int_0^{2\pi}\left(1-2\cos 2\theta+\cos^2 2\theta\right)d\theta$$

$$= \frac{2\pi}{4} + \frac{1}{4}\int_0^{2\pi} \frac{1+\cos 4\theta}{2}\, d\theta$$

$$= \frac{\pi}{2} + \frac{1}{4}\,\frac{1}{2}\,2\pi = \frac{3}{4}\pi.$$

$$\iint_{T_1} \vec{F}\cdot\hat{n}\, dA = -\frac{1}{6} r^6 \frac{3}{4}\pi = -\frac{1}{8}\pi r^6$$

$$\iint_{T_2} \vec{F}\cdot\hat{n}\, dA = -\iint_{T_2} z^4\, dA = -\frac{1}{8}\pi r^6$$ (es igual que la integral anterior)

Entonces

$$\iint_{\Sigma} \vec{F}\cdot\hat{n}\, dA = \pi^2 R r^2 + \frac{1}{8}\pi r^6 + \frac{1}{8}\pi r^6$$

$$= \pi^2 R r^2 + \frac{1}{4}\pi r^6.$$

3) $\Delta f = \operatorname{div}(\nabla f) = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$

<u>En cilíndricas</u>

$$x = \rho \cos\theta \qquad h_\rho = 1$$
$$y = \rho \operatorname{sen}\theta \qquad h_\theta = \rho$$
$$z = z \qquad h_z = 1$$

$$\operatorname{div}(F_\rho \hat{\rho} + F_\theta \hat{\theta} + F_z \hat{z}) = \frac{1}{\rho}\left(\frac{\partial}{\partial \rho}(\rho F_\rho) + \frac{\partial}{\partial \theta} F_\theta + \frac{\partial}{\partial z}(\rho F_z)\right)$$

$$\nabla f = \frac{\partial f}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{\partial f}{\partial z}\hat{z}$$

$$\Delta f = \frac{1}{\rho}\left(\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{\rho}\frac{\partial f}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(\rho \frac{\partial f}{\partial z}\right)\right)$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

<u>En coordenadas esféricas</u>

$$x = r \operatorname{sen}\phi \cos\theta \qquad h_r = 1$$
$$y = r \operatorname{sen}\phi \operatorname{sen}\theta \qquad h_\theta = r\operatorname{sen}\phi$$
$$z = r\cos\phi \qquad h_\phi = r$$

$$\operatorname{div}(F_r \hat{r} + F_\theta \hat{\theta} + F_\phi \hat{\phi}) = \frac{1}{r^2 \operatorname{sen}\phi}\left[\frac{\partial}{\partial r}(r^2 \operatorname{sen}\phi\, F_r) + \frac{\partial}{\partial \theta}(r F_\theta) + \frac{\partial}{\partial \phi}(r \operatorname{sen}\phi\, F_\phi)\right]$$

$$\nabla f = \frac{\partial f}{\partial r}\,\hat{r} + \frac{1}{r \operatorname{sen}\phi}\,\frac{\partial f}{\partial \theta}\,\hat{\theta} + \frac{1}{r}\,\frac{\partial f}{\partial \phi}\,\hat{\phi}$$

$$\Delta f = \frac{1}{r^2 \operatorname{sen}\phi}\left[\frac{\partial}{\partial r}\left(r^2 \operatorname{sen}\phi\,\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(r \cdot \frac{1}{r \operatorname{sen}\phi}\,\frac{\partial f}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\left(r \operatorname{sen}\phi\,\frac{1}{r}\,\frac{\partial f}{\partial \phi}\right)\right]$$

$$= \frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^2\,\frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \operatorname{sen}^2\phi}\,\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r^2}\,\frac{\partial^2 f}{\partial \phi^2}$$