Auxiliar 9

$\boxed{\text{SP}_1}$ (a) $P_2(x,y) = f(0,0) + \nabla f(0,0)^t \begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2}(x,y)\, f''(0,0) \begin{pmatrix} x \\ y \end{pmatrix}$.

Donde $f(0,0) = 0 \cdot \log(1) + \text{sen}(0) = 0$,

$$\frac{\partial f}{\partial x}(x,y) = \log(1+y) + \cos(x+y), \quad \frac{\partial f}{\partial y}(x,y) = \frac{x}{1+y} + \cos(x+y),$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = -\text{sen}(x+y), \quad \frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{1}{1+y} - \text{sen}(x+y), \quad \frac{\partial^2 f}{\partial y^2}(x,y) = \frac{-x}{(1+y)^2} - \text{sen}(x+y).$$

Así, $\nabla f(0,0)^t = (1,1)$, $f''(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (notar que f es C^2 entorno del $(0,0)$ y por ello se tiene el Teorema de Schwartz)

Luego,

$$P_2(x,y) = (1,1)\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2}(x,y)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$
$$= x+y + \frac{1}{2}(y,x)\begin{pmatrix} x \\ y \end{pmatrix} = x+y+\frac{1}{2}(xy+xy) = x+y+xy.$$

Como f es $C^\infty(\mathbb{R}\times(-1,\infty))$, f admite un desarrollo de Taylor de orden 2 en torno al $(0,0)$, y el resto está dado por:

$$R_3(x,y) = f(x,y) - P_2(x,y) = \frac{1}{3!}\sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\sum_{i_3=1}^{2} \frac{\partial^3 f}{\partial x_{i_1}\partial x_{i_2}\partial x_{i_3}}(\xi(x,y))\, x_{i_1}x_{i_2}x_{i_3},$$

con $\xi \in (0,1)$.

$$\Rightarrow R_3(x,y) = \frac{1}{6}\left(\frac{\partial^3 f}{\partial x^3}(\xi(x,y))x^3 + 3\frac{\partial^3 f}{\partial x^2 \partial y}(\xi(x,y))x^2y + 3\frac{\partial^3 f}{\partial x \partial y^2}(\xi(x,y))xy^2 + \frac{\partial^3 f}{\partial y^3}(\xi(x,y))y^3\right).$$

Calculemos las terceras derivadas

$$\frac{\partial^3 f}{\partial x^3}(x,y) = -\cos(x+y), \quad \frac{\partial^3 f}{\partial x^2 \partial y}(x,y) = -\cos(x+y),$$

$$\frac{\partial^3 f}{\partial x \partial y^2}(x,y) = \frac{-1}{(1+y)^2} - \cos(x+y), \quad \frac{\partial^3 f}{\partial y^3}(x,y) = \frac{2x}{(1+y)^3} - \cos(x+y).$$

Así,

$$R_3(x,y) = \frac{1}{6}\left(-\cos(\xi(x+y))x^3 - 3\cos(\xi(x+y))x^2y - \frac{3xy^2}{(1+\xi y)^2} - 3\cos(\xi(x+y))xy^2 + \frac{2xy^3\xi}{(1+\xi y)^3} - \cos(\xi(x+y))y^3\right)$$

$$\Rightarrow R_3(x,y) = \frac{1}{6}\left(-\cos(\xi(x+y))(x+y)^3 + \frac{2xy^3\xi - 3xy^2 - 3xy^3\xi}{(1+\xi y)^3}\right)$$

$$\Rightarrow R_3(x,y) = \frac{1}{6}\left(-\cos(\xi(x+y))(x+y)^3 - \frac{(3xy^2 + xy^3\xi)}{(1+\xi y)^3}\right).$$

Como $x^2+y^2 \le \frac{1}{4} \Rightarrow x^2 \le \frac{1}{4} \wedge y^2 \le \frac{1}{4} \Rightarrow |x| \le \frac{1}{2} \wedge |y| \le \frac{1}{2} \Rightarrow |x|\xi \le \frac{1}{2} \wedge |y|\xi \le \frac{1}{2}$,

luego,

$$|R_3(x,y)| \le \frac{1}{6}\left(|\cos(\xi(x+y))|\,(|x|+|y|)^3 + \frac{(3|x||y|^2 + |y|^3|x|\xi)}{(1-\xi|y|)^3}\right)$$

$$\le \frac{1}{6}\left(1\cdot(|x|+|y|)^3 + \frac{(3|x||y|^2 + \frac{|y|^3}{2})}{(1-\frac{1}{2})^3}\right)$$

$$\le \frac{1}{6}\left((|x|+|y|)^3 + 8(|x|+|y|)^3\right) = \frac{9}{6}(|x|+|y|)^3 = \frac{3}{2}(|x|+|y|)^3. \quad \square$$

(b) Como f es $C^\infty(\mathbb{R}^2)$, por el Teorema de Taylor tenemos que

$$f(x,y) - P_4(x,y) = R_5(x,y), \quad \text{con} \quad \lim_{(x,y)\to(0,0)} \frac{R_5(x,y)}{\|(x,y)\|^4} = 0,$$

donde $P_4(x,y)$ es el polinomio de Taylor de orden 4 en torno al $(0,0)$.

Luego

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - P_4(x,y)}{x^4+y^4} = \lim_{(x,y)\to(0,0)} \frac{f(x,y)-P_4(x,y)}{\|(x,y)\|^4}\cdot\frac{\|(x,y)\|^4}{x^4+y^4};$$

como

$$\frac{\|(x,y)\|^4}{x^4+y^4} = \frac{(x^2+y^2)^2}{x^4+y^4} = \frac{x^4+2x^2y^2+y^4}{x^4+y^4} = 1 + \frac{2x^2y^2}{x^4+y^4}, \text{ y además}$$

$$0 \le 2x^2y^2 \le x^4+y^4 \Rightarrow \frac{\|(x,y)\|^4}{x^4+y^4} \in [1,2],$$

luego como tenemos límite de una función que converge a cero por una acotada:

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - P_4(x,y)}{x^4+y^4} = 0 \quad \text{como se quería.}$$

Para calcular $P_4(x,y)$ podemos hacerlo de dos maneras, primero calculando la serie de la exponencial:

$$f(x,y) = xy\cdot e^{x^2+2y^2} = xy\left(1 + (x^2+2y^2) + \frac{(x^2+2y^2)^2}{2} + \dots + \frac{(x^2+2y^2)^n}{n!} + \dots\right)$$

$$= \underbrace{xy + x^3y + 2xy^3}_{P_4(x,y)} + \underbrace{xy\left(\frac{(x^2+2y^2)^2}{2} + \dots + \frac{(x^2+2y^2)^n}{n!} + \dots\right)}_{R_5(x,y)}.$$

Ahora calculemos $P_4(x,y)$ directamente:

$$P_4(x,y) = f(0,0) + \left(\frac{\partial f}{\partial x}(0,0)\cdot x + \frac{\partial f}{\partial y}(0,0)\cdot y\right) + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}(0,0)\,x^2 + 2\frac{\partial^2 f}{\partial x \partial y}(0,0)\,xy + \frac{\partial^2 f}{\partial y^2}(0,0)\,y^2\right) +$$

$$+ \frac{1}{6}\left(\frac{\partial^3 f}{\partial x^3}(0,0)\,x^3 + 3\frac{\partial^3 f}{\partial x^2 \partial y}(0,0)\,x^2y + 3\frac{\partial^3 f}{\partial x \partial y^2}(0,0)\,xy^2 + \frac{\partial^3 f}{\partial y^3}(0,0)\,y^3\right) +$$

$$+ \frac{1}{24}\left(\frac{\partial^4 f}{\partial x^4}(0,0)\,x^4 + 4\frac{\partial^4 f}{\partial x^3 \partial y}(0,0)\,x^3y + 6\frac{\partial^4 f}{\partial x^2 \partial y^2}(0,0)\,x^2y^2 + 4\frac{\partial^4 f}{\partial x \partial y^3}(0,0)\,xy^3 + \frac{\partial^4 f}{\partial y^4}(0,0)\,y^4\right).$$

Se observa del hecho que $f(x,y) = xy\cdot g(x,y) \Rightarrow f(0,0)=0,\ \frac{\partial f}{\partial x}(0,0)=0,\ \frac{\partial f}{\partial y}(0,0)=0,\ \frac{\partial^2 f}{\partial x^2}(0,0)=0,$ $\frac{\partial^2 f}{\partial y^2}(0,0)=0,\ \frac{\partial^3 f}{\partial x^3}(0,0)=0,\ \frac{\partial^3 f}{\partial y^3}(0,0)=0,\ \frac{\partial^4 f}{\partial x^4}(0,0)=0,\ \frac{\partial^4 f}{\partial y^4}(0,0)=0.$

$$\Rightarrow P_4(x,y) = \frac{\partial^2 f}{\partial x \partial y}(0,0)\cdot xy + \frac{1}{2}\left(\frac{\partial^3 f}{\partial x^2 \partial y}(0,0)\,x^2y + \frac{\partial^3 f}{\partial x \partial y^2}(0,0)\,xy^2\right) + \frac{1}{12}\left(2\frac{\partial^4 f}{\partial x^3 \partial y}(0,0)\,x^3y + 3\frac{\partial^4 f}{\partial x^2 \partial y^2}(0,0)\,x^2y^2 +\right.$$

$$\left. 2\frac{\partial^4 f}{\partial x \partial y^3}(0,0)\,xy^3\right).$$

Además,

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(x,y)\right) = \frac{\partial}{\partial x}\left(xe^{x^2+2y^2} + 4xy^2e^{x^2+2y^2}\right) = \frac{\partial}{\partial x}\left(e^{x^2+2y^2}(x+4xy^2)\right)$$

$$= 2xe^{x^2+2y^2}(x+4xy^2) + e^{x^2+2y^2}(1+4y^2)$$

$$= e^{x^2+2y^2}(1+2x^2+4y^2+8x^2y^2),$$

$$\frac{\partial^3 f}{\partial x^2 \partial y}(x,y) = 2x\,e^{x^2+2y^2}(1+2x^2+4y^2+8x^2y^2) + e^{x^2+2y^2}(4x+16xy^2)$$

$$= e^{x^2+2y^2}(6x+4x^3+24xy^2+16x^3y^2),$$

$$\frac{\partial^3 f}{\partial x \partial y^2}(x,y) = 4y\,e^{x^2+2y^2}(1+2x^2+4y^2+8x^2y^2) + e^{x^2+2y^2}(8y+16x^2y^3)$$

$$= e^{x^2+2y^2}(12y+8x^2y+16y^3+48x^2y^3),$$

$$\frac{\partial^4 f}{\partial x^3 \partial y}(x,y) = 2x\,e^{x^2+2y^2}(6x+4x^3+24xy^2+16x^3y^2) + e^{x^2+2y^2}(6+12x^2+24y^2+48x^2y^2)$$

$$= e^{x^2+2y^2}(6+24x^2+24y^2+8x^4+96x^2y^2+32x^4y^2),$$

$$\frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y) = 2x\,e^{x^2+2y^2}(12y+8x^2y+16y^3+48x^2y^3) + e^{x^2+2y^2}(16xy+96xy^3)$$

$$= e^{x^2+2y^2}(30xy+16x^3y+128xy^3+96x^3y^3),$$

$$\frac{\partial^4 f}{\partial x \partial y^3}(x,y) = 4y\, e^{x^2+2y^2}(12y+8x^2y+16y^3+48x^2y^3) + e^{x^2+2y^2}(12+8x^2+48y^2+144x^2y^2)$$

$$= e^{x^2+2y^2}(12+8x^2+96y^2+176x^2y^2+64y^4+192x^2y^4).$$

Así,

$$P_4(x,y) = 1\cdot xy + \frac{1}{2}(0\cdot x^2y + 0\cdot xy^2) + \frac{1}{12}(2\cdot 6\cdot x^3y + 3\cdot 0\cdot x^2y^2 + 2\cdot 12\cdot xy^3)$$

$$\Rightarrow P_4(x,y) = xy + \frac{1}{12}(12x^3y + 24xy^3) = xy + x^3y + 2xy^3. \quad \square$$

S_{P_2}

(a) $\nabla f(x,y)^t = (\cos x - 2y \operatorname{Sen} x,\ 2y + 2\cos x)$.

Luego, (x,y) es punto crítico ssi:

$$\cos x = 2y \operatorname{Sen} x \ \wedge\ 2y + 2\cos x = 0$$

$$\Leftrightarrow \cos x = 2y \operatorname{Sen} x \ \wedge\ \cos x = -y$$

$$\Leftrightarrow -y = 2y \operatorname{Sen} x \ \wedge\ \cos x = -y$$

$$\Leftrightarrow y(1+2\operatorname{Sen} x) = 0 \ \wedge\ \cos x = -y$$

$$\Leftrightarrow (y=0 \wedge \cos x = 0) \vee \left(\operatorname{Sen} x = -\tfrac{1}{2} \wedge y = -\cos x\right)$$

$$\Leftrightarrow (x,y) \in \left\{\left(\tfrac{(2k+1)}{2}\pi, 0\right) \mid k \in \mathbb{Z}\right\} \cup \left\{\left(\tfrac{7\pi}{6}+2k\pi, \tfrac{\sqrt{3}}{2}\right) \mid k \in \mathbb{Z}\right\} \cup \left\{\left(-\tfrac{\pi}{6}+2k\pi, -\tfrac{\sqrt{3}}{2}\right) \mid k \in \mathbb{Z}\right\}.$$

$$f''(x,y) = \begin{pmatrix} -\operatorname{Sen} x - 2y\cos x & -2\operatorname{Sen} x \\ -2\operatorname{Sen} x & 2 \end{pmatrix}.$$

1°) Si $(x,y) \in \left\{\left(\frac{(2k+1)}{2}\pi, 0\right) \mid k \in \mathbb{Z}\right\} \Rightarrow y=0,\ \cos x = 0,\ \operatorname{Sen} x \in \{1,-1\}$, luego

$-\operatorname{Sen} x - 2y\cos x \in \{-1, 1\}$ y $\det f''(x,y) = -2\operatorname{Sen} x - 4\operatorname{Sen}^2 x = -2\operatorname{Sen} x - 4 \in \{-6,-2\}$

luego si λ_1, λ_2 son los valores propios de $f''(x,y) \Rightarrow \lambda_1 \cdot \lambda_2 = \det f''(x,y) \in \{-6,-2\}$

luego uno es positivo y el otro negativo, es decir, (x,y) es punto silla.

2º) Si $(x,y)\in\left\{\left(\frac{7\pi}{6}+2k\pi,\frac{\sqrt{3}}{2}\right) \mid k\in\mathbb{Z}\right\}\cup\left\{\left(-\frac{\pi}{6}+2k\pi,-\frac{\sqrt{3}}{2}\right) \mid k\in\mathbb{Z}\right\} \Rightarrow \operatorname{sen} x=-\frac{1}{2} \wedge y=-\cos x\in\left\{-\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}\right\}$

$$\Rightarrow -\operatorname{sen} x-2y\cos x=-\operatorname{sen} x+2y^2=\frac{1}{2}+2\cdot\frac{3}{4}=2>0,$$

$$\det f''(x,y)=-2\operatorname{sen} x-4y\cos x-4\operatorname{sen}^2 x=1+4y^2-4\cdot\frac{1}{4}=4\cdot\frac{3}{4}=3>0;$$

luego $f''(x,y)$ es definida positiva y (x,y) es mínimo local estricto. □

(b) Como f es C^2 por el Teorema de Schwartz $a=\frac{\partial^2 f}{\partial x\partial y}(x_0,y_0)=\frac{\partial^2 f}{\partial y\partial x}(x_0,y_0)$,

y como es armónica $b=\frac{\partial^2 f}{\partial x^2}(x_0,y_0)=-\frac{\partial^2 f}{\partial y^2}(x_0,y_0)$, luego

$$f''(x_0,y_0)=\begin{pmatrix} b & a \\ a & -b\end{pmatrix}\Rightarrow \det f''(x_0,y_0)=-b^2-a^2\leq 0$$

como $\det f''(x_0,y_0)=\lambda_1\cdot\lambda_2$ con λ_1,λ_2 valores propios de $f''(x_0,y_0)$; y como (x_0,y_0) es extremo local $\Rightarrow \lambda_1$ y λ_2 tienen igual signo $\Rightarrow \lambda_1\cdot\lambda_2\geq 0$

$$\Rightarrow \det f''(x_0,y_0)=-(a^2+b^2)=0\Rightarrow a=b=0. \ \square$$

$\boxed{Sp_3}$

(a) $$\nabla f(x,y)^t=\left(y e^{-x^2-y^2}+xy e^{-x^2-y^2}\cdot(-2x),\ x e^{-x^2-y^2}+xy e^{-x^2-y^2}\cdot(-2y)\right)$$

$$=e^{-x^2-y^2}\left(y-2x^2y,\ x-2xy^2\right).$$

Luego (x,y) es punto crítico ssi:

$$y=2x^2y \ \wedge\ x=2xy^2$$

$$\Longleftrightarrow y(1-2x^2)=0 \ \wedge\ x(1-2y^2)=0$$

$$\Longleftrightarrow (x,y)=(0,0)\ \vee\ (x^2,y^2)=\left(\tfrac{1}{2},\tfrac{1}{2}\right)$$

$$\Longleftrightarrow (x,y)\in\left\{(0,0),\left(\sqrt{\tfrac{1}{2}},\sqrt{\tfrac{1}{2}}\right),\left(-\sqrt{\tfrac{1}{2}},\sqrt{\tfrac{1}{2}}\right),\left(\sqrt{\tfrac{1}{2}},-\sqrt{\tfrac{1}{2}}\right),\left(-\sqrt{\tfrac{1}{2}},-\sqrt{\tfrac{1}{2}}\right)\right\}.$$

$$f''(x,y)=\begin{pmatrix} -2xe^{-x^2-y^2}(y-2x^2y)+e^{-x^2-y^2}\cdot(-4xy) & -2xe^{-x^2-y^2}(x-2xy^2)+e^{-x^2-y^2}(1-2y^2) \\ -2xe^{-x^2-y^2}(x-2xy^2)+e^{-x^2-y^2}(1-2y^2) & -2ye^{-x^2-y^2}(x-2xy^2)+e^{-x^2-y^2}(-4xy)\end{pmatrix}$$

$$\Rightarrow f''(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad f''\left(\sqrt{\tfrac{1}{2}}, \sqrt{\tfrac{1}{2}}\right) = f''\left(-\sqrt{\tfrac{1}{2}}, -\sqrt{\tfrac{1}{2}}\right) = \begin{pmatrix} -\frac{2}{e} & 0 \\ 0 & -\frac{2}{e} \end{pmatrix},$$

$$f''\left(-\sqrt{\tfrac{1}{2}}, \sqrt{\tfrac{1}{2}}\right) = f''\left(\sqrt{\tfrac{1}{2}}, -\sqrt{\tfrac{1}{2}}\right) = \begin{pmatrix} \frac{2}{e} & 0 \\ 0 & \frac{2}{e} \end{pmatrix}.$$

Como $\det f''(0,0) = -1 < 0$, $(0,0)$ es punto silla (pues uno de sus valores propios es positivo y el otro es negativo, de hecho $p(t) = \det(f''(0,0) - t\,Id) = \det\begin{pmatrix} -t & 1 \\ 1 & -t \end{pmatrix} = t^2 - 1$, luego λ_1, λ_2 son las raíces de $p(t)$, que es el polinomio característico, así $\lambda_1 = 1, \lambda_2 = -1$).

Para $\left(\sqrt{\tfrac{1}{2}}, \sqrt{\tfrac{1}{2}}\right)$ y $\left(-\sqrt{\tfrac{1}{2}}, -\sqrt{\tfrac{1}{2}}\right)$ $f''(x,y)$ es definida negativa y por ello son máximos locales estrictos, y para $\left(-\sqrt{\tfrac{1}{2}}, \sqrt{\tfrac{1}{2}}\right)$ y $\left(\sqrt{\tfrac{1}{2}}, -\sqrt{\tfrac{1}{2}}\right)$ obtenemos $f''(x,y)$ definida positiva y por ello son mínimos locales estrictos. $\square$

(b)

$$0 \leq |f(x,y)| = \frac{|x||y|}{e^{x^2+y^2}} \leq \frac{x^2+y^2}{2e^{x^2+y^2}}, \quad \text{tomando el cambio } t = x^2+y^2:$$

$$0 \leq \lim_{\|(x,y)\| \to \infty} |f(x,y)| \leq \lim_{\sqrt{t} \to \infty} \frac{t}{2e^t} = \lim_{t \to \infty} \frac{t}{2e^t} = 0$$

$$\Rightarrow \lim_{\|(x,y)\| \to \infty} f(x,y) = 0.$$

Así, como $f\left(\sqrt{\tfrac{1}{2}}, \sqrt{\tfrac{1}{2}}\right) = f\left(-\sqrt{\tfrac{1}{2}}, -\sqrt{\tfrac{1}{2}}\right) = \frac{1}{2e} > 0$, $f\left(-\sqrt{\tfrac{1}{2}}, \sqrt{\tfrac{1}{2}}\right) = f\left(\sqrt{\tfrac{1}{2}}, -\sqrt{\tfrac{1}{2}}\right) = \frac{-1}{2e} < 0$ y $\lim_{\|(x,y)\| \to \infty} f(x,y) = 0$

entonces $\exists M > 0$ t.q. $\forall (x,y) \in \mathbb{R}^2$ t.q. $\|(x,y)\| > M \Rightarrow |f(x,y)| < \frac{1}{4e} < \frac{1}{2e}$,

luego como f es continua y $\overline{B(0,M)}$ es cerrado y acotado, f alcanza su máximo y mínimo en $\overline{B(0,M)}$ en puntos $(x^*,y^*), (x_*,y_*)$ respectivamente, luego $f(x^*,y^*) \geq f\left(\sqrt{\tfrac{1}{2}}, \sqrt{\tfrac{1}{2}}\right) = \frac{1}{2e} > f(x,y)$ $\forall (x,y) \notin \overline{B(0,M)}$; y $f(x_*,y_*) \leq f\left(-\sqrt{\tfrac{1}{2}}, \sqrt{\tfrac{1}{2}}\right) = \frac{-1}{2e} < f(x,y)$ $\forall (x,y) \notin \overline{B(0,M)}$. Así (x^*,y^*) y (x_*,y_*) son el máximo y mínimo de f en $\mathbb{R}^2$ y como pertenecen a $\overline{B(0,M)}$ son también máximo y mínimo local, luego deben ser puntos críticos, así, no queda otra opción que

$$(x^*,y^*) \in \left\{\left(\sqrt{\tfrac{1}{2}}, \sqrt{\tfrac{1}{2}}\right), \left(-\sqrt{\tfrac{1}{2}}, -\sqrt{\tfrac{1}{2}}\right)\right\}, \quad (x_*,y_*) \in \left\{\left(-\sqrt{\tfrac{1}{2}}, \sqrt{\tfrac{1}{2}}\right), \left(\sqrt{\tfrac{1}{2}}, -\sqrt{\tfrac{1}{2}}\right)\right\} \text{ y}$$

$$\max_{(x,y) \in \mathbb{R}^2} f(x,y) = f(x^*,y^*) = \frac{1}{2e}, \quad \min_{(x,y) \in \mathbb{R}^2} f(x,y) = f(x_*,y_*) = \frac{-1}{2e}. \quad \square$$