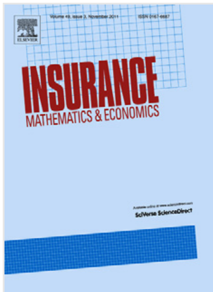

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Dynamic Hedging of Conditional Value-at-Risk*

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Abstract

In this paper the problem of partial hedging is studied by constructing hedging strategies that minimize conditional value-at-risk (CVaR) of the portfolio. Two dual versions of the problem are considered: minimization of CVaR with the initial wealth bounded from above, and minimization of hedging costs subject to a CVaR constraint. The Neyman-Pearson lemma approach is used to deduce semi-explicit solutions. Our results are illustrated by constructing CVaR-efficient hedging strategies for a call option in the Black-Scholes model and also for an embedded call option in an equity-linked life insurance contract.

JEL classification: C61, G13, G22.

Subject category: IM01, IM10, IM53.

Insurance branch category: IB10.

Keywords: conditional value-at-risk, dynamic hedging, stochastic modelling, quantile hedging, unit-linked contracts.

1 Introduction

In a complete financial market every contingent claim with payoff H delivered at time $t = T$ can be hedged perfectly: given the sufficient amount of the initial wealth, an agent who holds a short position in claim H can construct a portfolio (V_t, ξ_t) that will replicate the liability without risk, that is, $V_T =$

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H, a.s. On the other hand, an agent who is unwilling to invest in perfect replication but still aims to reduce the risk exposure, may address to partial hedging methods.

The problem of partial hedging is to construct a portfolio that minimizes the risk of the difference $L = H - V_T$. Efficiency and consistency of this approach depend to a great extent on selecting a specific way of quantifying the risk. For instance, one of the most studied methods known as quadratic hedging suggests minimizing quadratic error $\mathbb{E}[L^2]$. Despite its simplicity, this method has obvious disadvantages since the quadratic risk measure does not distinguish between loss and profit and equally penalizes both. Another method, known as quantile hedging, involves maximizing the probability of successful hedge $\mathbb{P}[L \leqslant 0]$ (see e.g. Föllmer & Leukert, 1999 or Cvitanić & Spivak, 1999). The Neyman-Pearson lemma (see Cvitanić & Karatzas, 2001) is exploited to derive quantile hedging strategies explicitly. This approach is generalized in Föllmer and Leukert (2000) to analyze the problem of minimizing expected loss $\mathbb{E}[L^+]$ and more generally, $\mathbb{E}[l(L^+)]$ for some loss function $l(\cdot)$.

We address the problem of partial hedging by minimizing conditional value-at-risk (CVaR), a quantile downside risk measure which is rapidly gaining popularity among risk professionals. Unlike risk measures mentioned above, CVaR (also known as expected shortfall or expected tail loss) is a coherent (Artzner, Delbaen, Eber, & Heath, 1999) and spectral (Acerbi, 2002) measure of risk. Aside from being a mathematically attractive tool, CVaR is also an economically consistent measure for hedging activity from the capital allocation point of view (see Goovaerts & Laeven, 2004 and Goovaerts, Kaas, & Laeven, 2010). Indeed, suppose that an investment company follows a risk management policy under which (i) CVaR is used as a measure of economic capital, (ii) all trading desks are using a unified approach to hedge against contingent claims they sell, (iii) risk monitoring is performed in a decentralized manner. In this case, adopting CVaR-minimizing hedging strategies implies achieving minimum economic capital requirements on a per claim basis.

Our main objective in this paper is to derive a hedging strategy which minimizes conditional value-at-risk of L subject to a constraint on the initial wealth; we also consider the dual problem: minimization of hedging costs subject to a constraint on CVaR. We suggest a method which can be used to construct CVaR-optimal hedging strategies explicitly in complete market models, which we illustrate by providing closed-form solutions for a call option in the Black-Scholes model and for an embedded option in a unit-linked life insurance contract. Note that a somewhat related problem was discussed in Li and Xu (2008) from portfolio optimization point of view: minimization of CVaR when the returns are bounded; in the present paper, however, we focus on derivatives hedging and insurance applications under capital constraints.

The paper is organized as follows. We start with defining conditional value-at-risk and describing the general probabilistic setup in Sections 2.1 and 2.2. In Section 2.3 we consider the problem of CVaR-efficient hedging under a capital constraint from contingent claim seller's point of view. With the help of CVaR optimization techniques mentioned in Rockafellar and Uryasev (2002) together with the Neyman-Pearson lemma approach suggested in Föllmer and Leukert (2000), we reduce our problem to a problem of one-dimensional optimization, which allows us to construct optimal hedges semi-explicitly. The dual problem is discussed in Section 2.4.

In Section 3 we illustrate our results by constructing CVaR-efficient hedging strategies for a call option in the classical Black-Scholes model first, and then minimizing costs of a hedging strategy subject to a CVaR constraint; numerical examples are presented for both problems.

In Section 4 we demonstrate how the quantile hedging methodology presented in Melnikov and Skornyakova (2005) and Melnikov and Romaniuk (2006) can be employed for CVaR hedging of an embedded call option in an equity-linked life insurance contract; the section is also concluded with numerical illustrations. We need to mention that CVaR is becoming a popular and efficient tool in modern actuarial science: refer, for instance, to Tan, Weng, and Zhang (2009) and Tian, Cox, Lin, and Zuluaga (2010).

To conclude the introductory section, the authors would like to thank an anonymous referee for valuable comments and suggestions which helped shape the final version of this paper.

2 CVaR Hedging

2.1 Conditional Value-at-Risk

Consider probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a choice-dependent $\mathcal{F}$-measurable random variable $L = L(x)$ characterizing the loss, with strategy vector $x \in X$ and strategy constraints X. We assume that $\mathbb{E}_{\mathbb{P}}[|L(x)|] < \infty$ for all $x \in X$.

Let $L_{(\alpha)}(x)$ and $L^{(\alpha)}(x)$ be lower and upper α-quantiles of $L(x)$ respectively:

$$L_{(\alpha)} = L_{(\alpha)}(x) = \inf\{t \in \mathbb{R} \ : \ \mathbb{P}[L \leqslant t] \geqslant \alpha\},$$
$$L^{(\alpha)} = L^{(\alpha)}(x) = \inf\{t \in \mathbb{R} \ : \ \mathbb{P}[L \leqslant t] > \alpha\}.$$

For a given strategy x and a fixed confidence level $\alpha \in (0, 1)$ which in applications would be a value fairly close to 1, value-at-risk (VaR) is defined as an upper α-quantile of the corresponding loss function,

$$\mathrm{VaR}^{\alpha}(L) = L^{(\alpha)}.$$

Conditional value-at-risk (CVaR), also known as expected shortfall, is defined as

$$\mathrm{CVaR}^{\alpha}(L) = \frac{1}{1-\alpha}\left(\mathbb{E}_{\mathbb{P}}\left[L \cdot \mathbf{1}_{\{L \geqslant L^{(\alpha)}\}}\right] + L^{(\alpha)}\left(1 - \alpha - \mathbb{P}\left[L \geqslant L^{(\alpha)}\right]\right)\right).$$

As shown in Acerbi and Tasche (2002), CVaR is closely related to the notion of tail conditional expectation (TCE). Indeed, in a smooth case, when

$$\mathbb{P}\left[L \geqslant L_{(\alpha)}\right] = 1 - \alpha, \qquad \mathbb{P}\left[L > L^{(\alpha)}\right] > 0,$$

or

$$\mathbb{P}\left[L \geqslant L_{(\alpha)},\ L \neq L^{(\alpha)}\right] = 0,$$

CVaR coincides with both upper and lower TCE:

$$\mathrm{TCE}_{\alpha}(L) = \mathbb{E}_{\mathbb{P}}\left[L \mid L \geqslant L^{(\alpha)}\right], \qquad \mathrm{TCE}^{\alpha}(L) = \mathbb{E}_{\mathbb{P}}\left[L \mid L \geqslant L_{(\alpha)}\right].$$

It is well-known that it is possible to compute both VaR and CVaR simultaneously by solving a certain one-dimensional convex optimization problem. This fact is presented in the theorem below; full text of the theorem may be found in Rockafellar and Uryasev (2002).

Let us define an auxiliary function

$$F_{\alpha}(x, z) = z + \frac{1}{1-\alpha} \cdot \mathbb{E}_{\mathbb{P}}\left[(L(x) - z)^{+}\right] \tag{2.1}$$

and note that for simplicity of notation we shall use $\mathrm{VaR}^\alpha(x)$, $\mathrm{CVaR}^\alpha(x)$ and $\mathrm{VaR}^\alpha(L(x))$, $\mathrm{CVaR}^\alpha(L(x))$ interchangeably.

Theorem 2.1 *As a function of z, function $F_\alpha(x,z)$ defined by (2.1) is finite and convex (hence continuous), and*

$$\mathrm{CVaR}_\alpha(x) = \min_{z\in\mathbb{R}} F_\alpha(x,z),$$
$$\mathrm{VaR}_\alpha(x) = \min\left\{y \mid y \in \mathrm{argmin}_{z\in\mathbb{R}} F_\alpha(x,z)\right\}.$$

In particular, one always has

$$\mathrm{VaR}_\alpha(x) \in \mathrm{argmin}_{z\in\mathbb{R}} F_\alpha(x,z),$$
$$\mathrm{CVaR}_\alpha(x) = F_\alpha(x, \mathrm{VaR}_\alpha(x)).$$

This theorem sheds some light on the question of why CVaR is a more stable performance criterion than VaR: it is well known in the optimization theory that the optimal value generally admits a more robust behavior than the argminimum.

An important corollary is that the problem of CVaR minimization may be expressed as a problem of $F_\alpha(x,z)$ minimization.

Corollary 2.2 *Minimization of $\mathrm{CVaR}_\alpha(x)$ over the strategy set X is equivalent to minimization of $F_\alpha(x,z)$ over $X\times\mathbb{R}$:*

$$\min_{x\in X} \mathrm{CVaR}_\alpha(x) = \min_{x\in X}\min_{z\in\mathbb{R}} F_\alpha(x,z).$$

2.2 General Setup

Let the discounted stock price be described by a stochastic process X_t on a standard stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ with $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

A *self-financing strategy* is defined by initial wealth $V_0 > 0$ and a predictable process ξ_t determining portfolio dynamics. For each strategy (V_0, ξ) the corresponding value process V_t is

$$V_t = V_0 + \int_0^t \xi_s dX_s, \quad \forall t \in [0,T].$$

We shall call a strategy (V_0, ξ) *admissible* if it satisfies

$$V_t \geqslant 0, \quad \forall t\in[0,T], \quad \mathbb{P}-a.s.,$$

and we shall denote the set of all admissible self-financing strategies by $\mathcal{A}$.

Consider a discounted contingent claim whose payoff is an $\mathcal{F}_T$–measurable non-negative random variable $H \in L^1(\mathbb{P})$. In a complete market there exists a unique equivalent martingale measure $\mathbb{P}^* \approx \mathbb{P}$, and construction of a perfect hedge is always possible. The perfect hedging strategy requires allocating the initial wealth in the amount of

$$H_0 = \mathbb{E}_{\mathbb{P}^*}[H].$$

The first question is: if, for some reason, it is impossible to allocate the required amount of initial wealth H_0 for hedging, what is the best hedge that can be constructed using a smaller amount $\tilde{V}_0 < H_0$? Evidently, perfect hedging cannot be used in this case; instead, we have access to an infinite set of partial hedges, and to come to determination we need to fix an optimality criterion. As such, conditional value-at-risk (CVaR) risk measure shall be used.

We define the loss function from the viewpoint of a claim seller who hedges a short position in H with portfolio (V_0, ξ), thus the loss at time T equals the claim value less the terminal value of the hedging portfolio:

$$L(V_0, \xi) = H - V_T = H - V_0 - \int_0^T \xi_s dX_s. \tag{2.2}$$

For a fixed confidence level $\alpha \in (0, 1)$, our first problem is to find an admissible strategy (V_0, ξ) which minimizes $\mathrm{CVaR}_\alpha(V_0, \xi)$ while using no more initial wealth than $\tilde{V}_0$.

Another question relates to the dual problem — what is the least amount of the initial wealth we have to allocate in order to keep CVaR of a given confidence level below a certain threshold? Again, it may be formulated as an optimization problem.

Both problems will be discussed in full in Sections 2.3 and 2.4.

2.3 Minimizing Conditional Value-at-Risk

In this section we suggest a method of solving the problem of CVaR minimization subject to a constraint on the initial wealth:

$$\begin{cases} \mathrm{CVaR}_\alpha(V_0, \xi) \longrightarrow \min\limits_{(V_0,\xi)\in\mathcal{A}}, \\ V_0 \leqslant \tilde{V}_0. \end{cases} \tag{2.3}$$

For simplicity of notation, denote by $\mathcal{A}_{\tilde{V}_0}$ the set of all admissible strategies satisfying the wealth constraint:

$$\mathcal{A}_{\tilde{V}_0} = \{(V_0, \xi) \mid (V_0, \xi) \in \mathcal{A}, \quad V_0 \leqslant \tilde{V}_0\}.$$

According to Corollary 2.1, problem (2.3) is equivalent to the following one:

$$F_\alpha((V_0, \xi), z) \longrightarrow \min_{(V_0,\xi)\in\mathcal{A}_{\tilde{V}_0}} \min_{z\in\mathbb{R}}.$$

Recall that F_α is given by (2.1) and that the loss function for this problem is given by (2.2), so the original problem becomes

$$z + \tfrac{1}{1-\alpha} \cdot \mathbb{E}_\mathbb{P}\left[(H - V_T - z)^+\right] \longrightarrow \min_{(V_0,\xi)\in\mathcal{A}_{\tilde{V}_0}} \min_{z\in\mathbb{R}}.$$

Let us introduce a function

$$c(z) = z + \tfrac{1}{1-\alpha} \cdot \min_{(V_0,\xi)\in\mathcal{A}_{\tilde{V}_0}} \mathbb{E}_\mathbb{P}\left[(H - V_T - z)^+\right], \tag{2.4}$$

such that

$$\min_{(V_0,\xi)\in\mathcal{A}_{\tilde{V}_0}} \mathrm{CVaR}_\alpha(V_0, \xi) = \min_{z\in\mathbb{R}} c(z).$$

Assume that for each $z \in \mathbb{R}$ the minimum in (2.4) is attained at $(\hat{V}_0(z), \hat{\xi}(z))$ and that $c(z)$ reaches its global minimum at point $\hat{z}$:

$$\min_{(V_0,\xi)\in\mathcal{A}_{\tilde{V}_0}} \mathbb{E}_\mathbb{P}\left[(H - V_T - z)^+\right] = \mathbb{E}_\mathbb{P}\left[(H - \hat{V}_T(z) - z)^+\right],$$
$$\min_{z\in\mathbb{R}} c(z) = c(\hat{z}).$$

Then strategy $(\hat{V}_0, \hat{\xi}) = (\hat{V}_0(\hat{z}), \hat{\xi}(\hat{z}))$ is a solution for (2.3):

$$\min_{(V_0,\xi)\in\mathcal{A}_{\tilde{V}_0}} \mathrm{CVaR}_\alpha(V_0, \xi) = \mathrm{CVaR}_\alpha(\hat{V}_0(\hat{z}), \hat{\xi}(\hat{z})).$$

Definition (2.4) of function $c(z)$ contains expected value minimization. Deriving explicit expression for this function would provide the possibility to reduce the initial problem (2.3) to a problem of one-dimensional optimization; to do that we shall use some known results in the area of expected shortfall minimization.

For each z strategy $(\hat{V}_0(z), \hat{\xi}(z))$ is a solution for the following problem:

$$\mathbb{E}_\mathbb{P}\left[(H - V_T - z)^+\right] \longrightarrow \min_{(V_0,\xi)\in\mathcal{A}_{\tilde{V}_0}}. \tag{2.5}$$

Note that

$$(H - V_T - z)^+ \equiv ((H - z)^+ - V_T)^+.$$

It is easy to see that $(H-z)^+$ is an $\mathcal{F}_T$–measurable, non-negative random variable — therefore we can consider it a contingent claim. Problem (2.5) then may be restated as

$$\mathbb{E}_{\mathbb{P}}\left[((H-z)^+ - V_T)^+\right] \longrightarrow \min_{(V_0,\xi)\in\mathcal{A}_{\tilde{V}_0}}. \tag{2.6}$$

Problem (2.6) can be treated as a problem of expected shortfall minimization with respect to a contingent claim with payoff $(H-z)^+$ that depends on a real-valued parameter z. This kind of problem has been well studied; we shall employ the results of Föllmer and Leukert (2000) to derive the solution.

Theorem 2.3 *The optimal strategy $(\hat{V}_0(z), \hat{\xi}(z))$ for problem (2.6) is a perfect hedge for the contingent claim $\tilde{H}(z) = (H-z)^+\tilde{\varphi}(z)$:*

$$\mathbb{E}_{\mathbb{P}^*}[\tilde{H}(z) \mid \mathcal{F}_t] = \hat{V}_0(z) + \int_0^t \hat{\xi}_s(z)dX_s, \quad \mathbb{P}\text{-a.s.}, \quad \forall t\in[0,T], \tag{2.7}$$

where

$$\tilde{\varphi}(z) = \mathbf{1}_{\left\{\frac{d\mathbb{P}}{d\mathbb{P}^*} > \tilde{a}(z)\right\}} + \gamma(z)\cdot\mathbf{1}_{\left\{\frac{d\mathbb{P}}{d\mathbb{P}^*} = \tilde{a}(z)\right\}}, \tag{2.8}$$

$$\tilde{a}(z) = \inf\left\{a \geqslant 0 : \mathbb{E}_{\mathbb{P}^*}\left[(H-z)^+\cdot\mathbf{1}_{\left\{\frac{d\mathbb{P}}{d\mathbb{P}^*} > a\right\}}\right] \leqslant \tilde{V}_0\right\}, \tag{2.9}$$

$$\gamma(z) = \frac{\tilde{V}_0 - \mathbb{E}_{\mathbb{P}^*}\left[(H-z)^+\cdot\mathbf{1}_{\left\{\frac{d\mathbb{P}}{d\mathbb{P}^*} > \tilde{a}(z)\right\}}\right]}{\mathbb{E}_{\mathbb{P}^*}\left[(H-z)^+\cdot\mathbf{1}_{\left\{\frac{d\mathbb{P}}{d\mathbb{P}^*} = \tilde{a}(z)\right\}}\right]}. \tag{2.10}$$

Theorem 2.3 provides an explicit solution for (2.6) in terms of the Neyman-Pearson framework – that is, $\tilde{\varphi}(z)$ may be interpreted as an optimal randomized test. Note that $\gamma(z)$ equals zero if the distribution of the Radon-Nikodym derivative $\frac{d\mathbb{P}}{d\mathbb{P}^*}$ is atomless.

Let us summarize the results of this section in the following theorem.

Theorem 2.4 *The optimal strategy $(\hat{V}_0, \hat{\xi})$ for the problem of CVaR minimization (2.3) is a perfect hedge for the contingent claim $\tilde{H}(\hat{z}) = (H-\hat{z})^+\tilde{\varphi}(\hat{z})$, where $\tilde{\varphi}(z)$ is defined by (2.8)-(2.10), $\hat{z}$ is a point of global minimum of function*

$$c(z) = \begin{cases} z + \frac{1}{1-\alpha}\mathbb{E}_{\mathbb{P}}\left[(H-z)^+(1-\tilde{\varphi}(z))\right], & \text{for } z < z^*, \\ z, & \text{for } z \geqslant z^*, \end{cases} \tag{2.11}$$

on interval $z < z^*$, and z^* is a real root of equation

$$\tilde{V}_0 = \mathbb{E}_{\mathbb{P}^*}[(H-z^*)^+].$$

Besides, one always has

$$\mathrm{CVaR}_\alpha(\hat{V}_0, \hat{\xi}) = c(\hat{z}), \tag{2.12}$$

$$\mathrm{VaR}_\alpha(\hat{V}_0, \hat{\xi}) = \hat{z}. \tag{2.13}$$

We used the results of Theorem 2.3 to get rid of the minimum in the definition of $c(z)$. Note that problem (2.6) only makes sense when $\tilde{V}_0 < \mathbb{E}_{\mathbb{P}^*}[H(z)]$, otherwise a perfect hedge for $H(z)$ may be used as an optimal strategy, providing zero expected shortfall. As a function of z, $\mathbb{E}_{\mathbb{P}^*}[H(z)]$ is monotonous and non-increasing, and

$$\mathbb{E}_{\mathbb{P}^*}[H(0)] = H_0 > \tilde{V}_0,$$
$$\lim_{z\to\infty} \mathbb{E}_{\mathbb{P}^*}[H(z)] = 0,$$

so there exists $z^* > 0$ such that

$$\tilde{V}_0 \geqslant \mathbb{E}_{\mathbb{P}^*}[H(z)], \quad \forall z \geqslant z^*. \tag{2.14}$$

Hence, when z is greater than z^*, the perfect hedge for $(H - z)^+$ can be used in problem (2.6) — this explains why $c(z) = z$ for $z \geqslant z^*$.

According to Theorem 2.1, argminimum of $c(z)$ coincides with the value-at-risk of the CVaR-optimal hedge. Note that the loss function is always non-negative,

$$L(z) = H - \tilde{H}(z) = H - \tilde{\varphi}(z)(H - z)^+ \geqslant 0,$$

therefore the corresponding value-at-risk would be also non-negative, so $\hat{z} > 0$; besides, $c(z) = z$ for $z > z^*$ and $c(z)$ is increasing at $z = z^*$, so the global minimum of $c(z)$ coincides with its minimum on $(0, z^*)$.

2.4 Minimizing Hedging Costs

In this section we minimize the initial wealth over all admissible strategies (V_0, ξ) with conditional value-at-risk of a given confidence level not exceeding predefined threshold $\tilde{C}$:

$$\begin{cases} V_0 \longrightarrow \min\limits_{(V_0,\xi)\in\mathcal{A}}, \\ \mathrm{CVaR}_\alpha(V_0, \xi) \leqslant \tilde{C}. \end{cases} \tag{2.15}$$

Let us rephrase the problem in terms of terminal capital $V_T = V_0 + \int_0^T \xi_s dX_s$ (we can always go back and derive the trading strategy explicitly by constructing a perfect hedge):

$$\begin{cases} \mathbb{E}_{\mathbb{P}^*}[V_T] \longrightarrow \min\limits_{V_T \in \mathcal{F}_T}, \\ \text{CVaR}_\alpha(V_T) \leqslant \tilde{C}. \end{cases} \tag{2.16}$$

Recall that

$$\text{CVaR}_\alpha(V_0, \xi) = \min_{z \in \mathbb{R}} \left(z + \frac{1}{1-\alpha} \mathbb{E}_{\mathbb{P}} \left[(H - V_T - z)^+ \right] \right),$$

and consider a family of problems

$$\begin{cases} \mathbb{E}_{\mathbb{P}^*}[V_T] \longrightarrow \min\limits_{V_T \in \mathcal{F}_T}, \\ \mathbb{E}_{\mathbb{P}}\left[(H - V_T - z)^+\right] \leqslant (\tilde{C} - z)(1-\alpha). \end{cases} \tag{2.17}$$

For consistency of notation, we provide the following lemma, which will be applied to problem (2.17).

Lemma 2.5 *Let $\tilde{x}$ be a solution for*

$$\begin{cases} f(x) \longrightarrow \min\limits_{x \in \mathbb{X}}, \\ \min\limits_{z \in \mathbb{R}} g(x, z) \leqslant c. \end{cases}$$

Then the following family of problems also admits solutions, denoted $\tilde{x}(z)$:

$$\begin{cases} f(x) \longrightarrow \min\limits_{x \in \mathbb{X}}, \\ g(x, z) \leqslant c. \end{cases}$$

Besides, one always has

$$\tilde{x} = \tilde{x}(\tilde{z}),$$

where z is a point of global minimum of $f(\tilde{x}(z))$:

$$\min_{z \in \mathbb{R}} f(\tilde{x}(z)) = f(\tilde{x}(\tilde{z})).$$

Indeed, for each $z \in \mathbb{R}$:

$$\bigcup_{z \in \mathbb{R}} \{x \mid g(x, z) \leqslant c\} = \left\{x \mid \min_{z \in \mathbb{R}} g(x, z) \leqslant c\right\},$$

and

$$\bigcup_{z \in \mathbb{R}} [\mathbb{X} \cap \{x \mid g(x, z) \leqslant c\}] = \mathbb{X} \cap \left\{x \mid \min_{z \in \mathbb{R}} g(x, z) \leqslant c\right\}.$$

Therefore,

$$\min_{x\in\mathbb{X}\cap\{x|\min\limits_{z\in\mathbb{R}} g(x,z)\leqslant c\}} f(x) = \min_{z\in\mathbb{R}}\left[\min_{x\in\mathbb{X}\cap\{x|g(x,z)\leqslant c\}} f(x)\right],$$

which proves the lemma. □

Denote the solution for (2.17) for each real z by $\tilde{V}_T(z)$, then, according to the lemma stated above, the solution for (2.16) may be expressed as

$$\tilde{V}_T = \tilde{V}_T(\tilde{z}), \tag{2.18}$$

where

$$\mathbb{E}_{\mathbb{P}^*}[\tilde{V}_T(\tilde{z})] = \min_{z\in\mathbb{R}} \mathbb{E}_{\mathbb{P}^*}[\tilde{V}_T(z)]. \tag{2.19}$$

We shall derive $\tilde{V}_T(z)$ by solving (2.17). To start with, note that in case $z > c$ the problem admits no solution since the left side is always non-negative.

In case $z \leqslant c$ note that

$$(H - V_T - z)^+ = ((H - z)^+ - V_T)^+,$$

and, since $0 \leqslant V_T \leqslant (H - z)^+$, let

$$V_T = (H - z)^+(1 - \varphi), \quad \varphi \in \mathcal{P}_{[0,1]},$$

where $\mathcal{P}_{[0,1]}$ is the class of $\mathcal{F}_T$-measurable random variables taking on values in $[0, 1]$. The initial problem can be then rewritten in terms of φ (its solution we will denote by $\tilde{\varphi}(z)$):

$$\begin{cases} \mathbb{E}_{\mathbb{P}}[(H - z)^+\varphi] \leqslant (\tilde{C} - z)(1 - \alpha), \\ \mathbb{E}_{\mathbb{P}^*}[(H - z)^+\varphi] \longrightarrow \max\limits_{\varphi\in\mathcal{P}_{[0,1]}}. \end{cases}$$

This problem can be solved by applying the Neyman-Pearson lemma, and it only makes sense as long as $\mathbb{E}_{\mathbb{P}}[(H - z)^+] > (\tilde{C} - z)(1 - \alpha)$, otherwise we can set $\tilde{\varphi}(z) \equiv 1$ and $\tilde{V}_T(z) \equiv 0$.

Lemma 2.6 *Condition*

$$\mathbb{E}_{\mathbb{P}}[(H - z)^+] > (\tilde{C} - z)(1 - \alpha) \tag{2.20}$$

is satisfied for all $z \leqslant \tilde{C}$ if and only if both of the following inequalities hold true:

$$\mathbb{E}_{\mathbb{P}}[H] > \tilde{C}(1 - \alpha), \qquad \mathbb{E}_{\mathbb{P}}[(H - \tilde{C})^+] > 0. \tag{2.21}$$

Note that both right- and left-hand sides of (2.20) are monotonous non-increasing functions of z. In addition,

$$\frac{d}{dz}\mathbb{E}_{\mathbb{P}}[(H-z)^+] = -1, \qquad \text{for } z<0,$$
$$\frac{d}{dz}(\tilde{C}-z)(1-\alpha) = -1+\alpha,$$

so it is necessary and sufficient that (2.20) holds true at points $z=0$ and $z=\tilde{C}$ only, which implies (2.21) and thus proves the lemma. □

Lemma 2.6 provides an easy way to check whether (2.20) is satisfied for all $z \leqslant \tilde{C}$ or not: if there exists such $z=z^*$ that it doesn't hold true, then $\tilde{V}_T(z^*) \equiv 0$ and, according to (2.18) and (2.19), the solution for (2.16) would be also equal to zero, which can be interpreted as selecting a passive trading strategy. Indeed, if the first inequality in (2.21) is not satisfied, the target CVaR is too high compared to the expected payoff on the contingent claim, so there is no need to hedge at all; if the second inequality is not satisfied, the payoff is bounded from above by a constant less than $\tilde{C}$, so CVaR can never reach its target value no matter what strategy is used.

Theorem 2.7 *The optimal strategy $(\hat{V}_0, \hat{\xi})$ for the problem of hedging costs minimization (2.15) is*

a) *a perfect hedge for the contingent claim $(H-\hat{z})^+(1-\tilde{\varphi}(\hat{z}))$ if condition (2.21) holds true, where $\tilde{\varphi}(z)$ is defined by*

$$\tilde{\varphi}(z) = \mathbf{1}_{\left\{\frac{d\mathbb{P}^*}{d\mathbb{P}} > \tilde{a}(z)\right\}} + \gamma(z)\cdot \mathbf{1}_{\left\{\frac{d\mathbb{P}^*}{d\mathbb{P}} = \tilde{a}(z)\right\}},$$
$$\tilde{a}(z) = \inf\left\{a \geqslant 0 : \mathbb{E}_{\mathbb{P}}\left[(H-z)^+\cdot \mathbf{1}_{\left\{\frac{d\mathbb{P}^*}{d\mathbb{P}} > a\right\}}\right] \leqslant (\tilde{C}-z)(1-\alpha)\right\},$$
$$\gamma(z) = \frac{(\tilde{C}-z)(1-\alpha) - \mathbb{E}_{\mathbb{P}}\left[(H-z)^+\cdot \mathbf{1}_{\left\{\frac{d\mathbb{P}^*}{d\mathbb{P}} > \tilde{a}(z)\right\}}\right]}{\mathbb{E}_{\mathbb{P}}\left[(H-z)^+\cdot \mathbf{1}_{\left\{\frac{d\mathbb{P}^*}{d\mathbb{P}} = \tilde{a}(z)\right\}}\right]},$$

and $\hat{z}$ is a point of minimum of function

$$d(z) = \mathbb{E}_{\mathbb{P}^*}\left[(H-z)^+(1-\tilde{\varphi}(z))\right]$$

on interval $-\infty < z \leqslant \tilde{C}$;

b) *a passive trading strategy if condition (2.21) is not satisfied.*

3 CVaR Hedging in the Black-Scholes Model

3.1 General Setup

In the framework of the standard Black-Scholes model, price of the underlying S_t and bond price B_t follow

$$\begin{cases} B_t = e^{rt}, \\ S_t = S_0 \exp(\sigma W_t + \mu t), \end{cases}$$

where r is riskless interest rate, $\sigma > 0$ is constant volatility, μ is constant drift and W is a Wiener process under $\mathbb{P}$. We assume there are no transaction costs and both instruments are freely tradable.

The SDE for the discounted price process $X_t = B_t^{-1} S_t$ is then given by

$$\begin{cases} dX_t = X_t(\sigma dW_t + m dt), \\ X_0 = x_0, \end{cases}$$

where $m = \mu - r + \frac{1}{2}\sigma^2$.

The unique equivalent martingale measure $\mathbb{P}^*$ may be derived with the help of the Girsanov theorem:

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp\left(-\frac{m}{\sigma} W_T - \frac{1}{2}\left(\frac{m}{\sigma}\right)^2 T\right), \tag{3.1}$$

Note that

$$X_T = x_0 \exp\left(\sigma W_T + (m - \tfrac{1}{2}\sigma^2)T\right),$$

so (3.1) may be rewritten as

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \text{const} \cdot X_T^{-m/\sigma^2}. \tag{3.2}$$

The contingent claim of interest in this section is a plain vanilla call option with payoff $(S_T - K)^+$. The discounted claim is also a call option with respect to X_t, with strike price of Ke^{-rT}:

$$H = (X_T - Ke^{-rT})^+,$$

Amount of the initial wealth H_0 required for a perfect hedge is

$$H_0 = \mathbb{E}_{\mathbb{P}^*}[H] = x_0 \Phi_+(Ke^{-rT}) - Ke^{-rT}\Phi_-(Ke^{-rT}),$$

where

$$\Phi_{\pm}(K) = \Phi\left(\frac{\ln x_0 - \ln K}{\sigma\sqrt{T}} \pm \frac{1}{2}\sigma\sqrt{T}\right),$$

and $\Phi(\cdot)$ is a cumulative distribution function for standard normal distribution.

3.2 Minimizing Conditional Value-at-Risk

In case the initial wealth is bounded above by $\tilde{V}_0 < H_0$, we cannot construct a perfect hedge for the call option; instead, we shall minimize CVaR over all admissible strategies with the initial wealth not exceeding $\tilde{V}_0$. The results of Section 2.3 shall be used to derive the explicit solution.

As stated in Theorem 2.4, the original problem may be reduced to a problem of minimizing an auxiliary function $c(z)$ on interval $(0, z^*)$, where

$$c(z) = \begin{cases} z + \frac{1}{1-\alpha}\mathbb{E}_{\mathbb{P}}\left[(H-z)^+\tilde{\varphi}(z)\right], & \text{for } z < z^*, \\ z, & \text{for } z \geqslant z^*, \end{cases}$$

$\tilde{\varphi}(z)$ is defined by (2.8)-(2.10) and z^* is a real root of

$$\tilde{V}_0 = \mathbb{E}_{\mathbb{P}^*}[(H - z^*)^+]. \tag{3.3}$$

Since we consider $z > 0$ only,

$$(H - z)^+ = ((X_T - Ke^{-rT})^+ - z)^+ = (X_T - (Ke^{-rT} + z))^+.$$

For simplicity of notation, denote

$$\begin{aligned} H(z) &= (X_T - K(z))^+, \\ K(z) &= Ke^{-rT} + z, \\ \tilde{\Phi}_{\pm}(x) &= \Phi_{\pm}\left(xe^{-mT}\right), \\ \Lambda_{\pm}(x,y) &= \Phi_{\pm}(x) - \Phi_{\pm}(y), \\ \tilde{\Lambda}_{\pm}(x,y) &= \tilde{\Phi}_{\pm}(x) - \tilde{\Phi}_{\pm}(y). \end{aligned}$$

It is clear that $H(z)$ is also a call option, with the strike price of $K(z)$, so we can apply the Black-Scholes formula to (3.3):

$$\tilde{V}_0 = x_0\Phi_+(K(z^*)) - K(z^*)\Phi_-(K(z^*)). \tag{3.4}$$

Further on, we shall refer to z^* as to the solution for (3.4).

We shall consider two cases.

(a) $\boldsymbol{\mu}+\frac{1}{2}\boldsymbol{\sigma}^2>\mathbf{r}$ $\quad(\mathbf{m}>\mathbf{0})$

The set $\left\{\frac{d\mathbb{P}}{d\mathbb{P}^*}>a\right\}$ takes form

$$\left\{\frac{d\mathbb{P}}{d\mathbb{P}^*}>a\right\}=\left\{X_T^{m/\sigma^2}>\hat{b}\right\}=\{X_T>b\},$$

and, moreover,

$$\mathbb{P}\left(\frac{d\mathbb{P}}{d\mathbb{P}^*}=a\right)=\mathbb{P}^*\left(\frac{d\mathbb{P}}{d\mathbb{P}^*}=a\right)=0.$$

Applying this to (2.8)–(2.10), we get

$$\begin{aligned}
\tilde{\varphi}(z) &= \mathbf{1}_{\left\{X_T>\tilde{b}(z)\right\}},\\
\tilde{b}(z) &= \inf\left\{b\geqslant 0:\mathbb{E}_{\mathbb{P}^*}\left[H(z)\cdot\mathbf{1}_{\{X_T>b)\}}\right]\leqslant\tilde{V}_0\right\},\\
\gamma(z) &= 0.
\end{aligned}$$

Note that in our case the infimum is always attained since we deal with atomless measures. The expectation in the expression for $\tilde{b}(z)$ may be rewritten as

$$\mathbb{E}_{\mathbb{P}^*}\left[H(z)\cdot\mathbf{1}_{\{X_T>b\}}\right]=\begin{cases}\mathbb{E}_{\mathbb{P}^*}[H(z)], & \text{for } b<K(z),\\ \\ x_0\Phi_+(b)-K(z)\Phi_-(b), & \text{for } b\geqslant K(z).\end{cases}$$

Since we consider $z<z^*$, (2.14) applies:

$$\mathbb{E}_{\mathbb{P}^*}[H(z)]>\tilde{V}_0,$$

therefore the minimum is not attained on the set $b<K(z)$, and hence $\tilde{b}(z)$ is a solution for the following system:

$$\begin{cases}x_0\Phi_+(b)-K(z)\Phi_-(b)=\tilde{V}_0,\\ \\ b\geqslant K(z).\end{cases}\tag{3.5}$$

Note that the constraint in (3.5) is essential since the equation may have more than one real root; it is straightforward to show that for all $0\leqslant z<z^*$ this system yields a single root $\tilde{b}(z)$.

Now we are able to write down the function $c(z)$:

$$c(z)=z+\tfrac{1}{1-\alpha}\cdot\mathbb{E}_{\mathbb{P}}\left[\mathbf{1}_{\left\{X_T\leqslant\tilde{b}(z)\right\}}\cdot H(z)\right],$$

or, evaluating the expectation,

$$c(z) = z + \tfrac{1}{1-\alpha} \cdot \left(x_0 e^{mT} \tilde{\Lambda}_+(K(z), \tilde{b}(z)) - K(z)\tilde{\Lambda}_-(K(z), \tilde{b}(z)) \right),$$

where $\tilde{b}(z)$ is a solution for (3.5).

According to Theorem 2.4, the optimal strategy $(\hat{V}_0, \hat{\xi})$ is then a perfect hedge for the contingent claim

$$\tilde{H}(\hat{z}) = (H - \hat{z})^+ \mathbf{1}_{\left\{X_T > \tilde{b}(\hat{z})\right\}},$$

where $\hat{z}$ is a point of minimum of $c(z)$ on interval $(0, z^*)$.

(b) $\boldsymbol{\mu}+\frac{1}{2}\boldsymbol{\sigma}^2< \mathbf{r} \quad (\mathbf{m} < \mathbf{0})$

In this case the set $\left\{\frac{d\mathbb{P}}{d\mathbb{P}^*} > a\right\}$ is

$$\left\{\frac{d\mathbb{P}}{d\mathbb{P}^*} > a\right\} = \left\{X_T^{m/\sigma^2} > \hat{b}\right\} = \{X_T < b\},$$

and therefore

$$\begin{aligned} \tilde{\varphi}(z) &= \mathbf{1}_{\left\{X_T < \tilde{b}(z)\right\}}, \\ \tilde{b}(z) &= \sup\left\{b \geqslant 0 : \mathbb{E}_{\mathbb{P}^*}\left[H(z) \cdot \mathbf{1}_{\{X_T < b)\}}\right] \leqslant \tilde{V}_0\right\}, \\ \gamma(z) &= 0. \end{aligned}$$

Denote

$$\beta(b, z) = x_0 \Lambda_+(K(z), b) - K(z)\Lambda_-(K(z), b),$$

then

$$\mathbb{E}_{\mathbb{P}^*}\left[H(z) \cdot \mathbf{1}_{\{X_T < b\}}\right] = \begin{cases} 0, & \text{for } b < K(z), \\ \beta(b, z), & \text{for } b \geqslant K(z). \end{cases}$$

Same as above, recall that $\mathbb{E}_{\mathbb{P}^*}[H(z)] > \tilde{V}_0$ for $z < z^*$, and consider properties of $\beta(b, z)$:

$$\beta(K(z), z) = 0, \qquad \beta(+\infty, z) = \mathbb{E}_{\mathbb{P}^*}[H(z)], \qquad \frac{\partial}{\partial b}\beta(b, z) \geqslant 0.$$

It is clear that the supremum is attained on the set $b \geqslant K(z)$, hence $\tilde{b}(z)$ is a solution (which exists and is unique) for the following system:

$$\begin{cases} x_0 \Lambda_+(K(z), b) - K(z)\Lambda_-(K(z), b) = \tilde{V}_0, \\ b \geqslant K(z). \end{cases} \tag{3.6}$$

Function $c(z)$ then takes form

$$c(z) = z + \tfrac{1}{1-\alpha} \cdot \mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{\left\{X_T \geqslant \tilde{b}(z)\right\}} \cdot H(z) \right],$$

or

$$c(z) = z + \tfrac{1}{1-\alpha} \cdot \left(x_0 e^{mT} \tilde{\Phi}_+(\tilde{b}(z)) - K(z) \tilde{\Phi}_-(\tilde{b}(z)) \right),$$

where $\tilde{b}(z)$ is a solution for (3.6) and the optimal strategy $(\hat{V}_0, \hat{\xi})$ is a perfect hedge for the contingent claim

$$\tilde{H}(\hat{z}) = (H - \hat{z})^+ \mathbf{1}_{\left\{X_T < \tilde{b}(\hat{z})\right\}},$$

where $\hat{z}$ is a point of minimum of $c(z)$ on interval $(0, z^*)$.

To illustrate the method numerically, consider a financial market that evolves in accordance with the Black-Scholes model with parameters $\sigma = 0.3$, $\mu = 0.09$, $r = 0.05$ and a plain vanilla call option with strike price of $K = 110$ and time to maturity $T = 0.25$. Let the initial price of the underlying be equal to $S_0 = 100$. In this setting, we are interested in hedging strategies that minimize $\mathrm{CVaR}_{0.975}$ (conditional value-at-risk with confidence level of 97.5%) for various amounts of the initial wealth. You can observe the results of numeric computations in Figure 1.

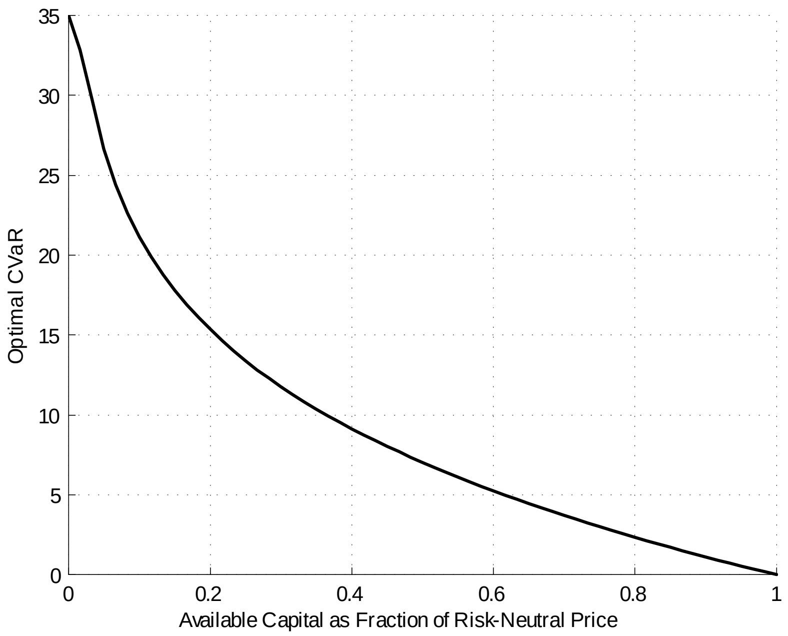

Figure 1. Minimizing CVaR for varying levels of the initial wealth.

3.3 Minimizing Hedging Costs

In this section we shall apply results of Section 2.4 to explicitly construct strategies minimizing the initial wealth with CVaR not exceeding target value $\tilde{C}$.

According to Theorem 2.7, a passive trading strategy is optimal in hedging costs minimization problem if at least one of inequalities (2.21) is not satisfied. In the Black-Scholes setting, these inequalities take form

$$\begin{aligned} &x_0 e^{mT}\tilde{\Phi}_+(K) - K\tilde{\Phi}_-(K) - \tilde{C}(1-\alpha) > 0, \\ &x_0 e^{mT}\tilde{\Phi}_+(K+\tilde{C}) - (K+\tilde{C})\tilde{\Phi}_-(K+\tilde{C}) > 0. \end{aligned} \tag{3.7}$$

Further on in this section we assume a non-trivial case, i.e. both inequalities in (3.7) are satisfied. Again, we consider two cases.

(a) $\boldsymbol{\mu}+\frac{1}{2}\boldsymbol{\sigma}^2>\mathbf{r}$ $\quad(\mathbf{m}>\mathbf{0})$

In this case

$$\left\{\frac{d\mathbb{P}^*}{d\mathbb{P}} > a\right\} = \{X_T < b\}, \qquad \mathbb{P}\left(\frac{d\mathbb{P}}{d\mathbb{P}^*} = a\right) = \mathbb{P}^*\left(\frac{d\mathbb{P}}{d\mathbb{P}^*} = a\right) = 0,$$

so we have

$$\begin{aligned} \tilde{\varphi}(z) &= \mathbf{1}_{\left\{X_T<\tilde{b}(z)\right\}}, \\ \tilde{b}(z) &= \sup\left\{b \geqslant 0 : \mathbb{E}_{\mathbb{P}}\left[H(z)\cdot \mathbf{1}_{\{X_T<b)\}}\right] \leqslant (\tilde{C}-z)(1-\alpha)\right\}, \\ \gamma(z) &= 0. \end{aligned}$$

Denote

$$\delta(b,z) = x_0 e^{mT}\tilde{\Lambda}_+(K(z),b) - K(z)\tilde{\Lambda}_-(K(z),b),$$

then

$$\mathbb{E}_{\mathbb{P}}\left[H(z)\cdot \mathbf{1}_{\{X_T<b\}}\right] = \begin{cases} 0, & \text{for } b < K(z), \\ \delta(b,z), & \text{for } b \geqslant K(z), \end{cases}$$

$$\delta(K(z),z) = 0, \qquad \delta(+\infty,z) = \mathbb{E}_{\mathbb{P}}\left[H(z)\right], \qquad \frac{\partial}{\partial b}\delta(b,z) \geqslant 0.$$

Assuming that inequalities (3.7) hold true, the supremum is attained on the set $b \geqslant K(z)$, hence $\tilde{b}(z)$ is a unique solution for the system

$$\begin{cases} x_0 e^{mT}\tilde{\Lambda}_+(K(z),b) - K(z)\tilde{\Lambda}_-(K(z),b) = (\tilde{C}-z)(1-\alpha), \\ b \geqslant K(z). \end{cases} \tag{3.8}$$

The optimal strategy $(\hat{V}_0, \hat{\xi})$ would be a perfect hedge for the contingent claim $H(\hat{z})^+ \mathbf{1}_{\left\{X_T > \tilde{b}(\hat{z})\right\}}$, where $\tilde{b}(z)$ is a solution for (3.8) and $\hat{z}$ is a point of minimum of function

$$d(z) = x_0 \Phi_+(\tilde{b}(z)) - K(z)\Phi_-(\tilde{b}(z))$$

on interval $z \in (-\infty, \tilde{C})$.

(b) $\boldsymbol{\mu}+\frac{1}{2}\boldsymbol{\sigma}^2< \mathbf{r} \quad (\mathbf{m} < \mathbf{0})$

We have

$$\left\{ \frac{d\mathbb{P}^*}{d\mathbb{P}} > a \right\} = \left\{ X_T > b \right\}, \qquad \mathbb{P}\left(\frac{d\mathbb{P}}{d\mathbb{P}^*} = a \right) = \mathbb{P}^* \left(\frac{d\mathbb{P}}{d\mathbb{P}^*} = a \right) = 0,$$

hence

$$\begin{aligned} \tilde{\varphi}(z) &= \mathbf{1}_{\left\{X_T > \tilde{b}(z)\right\}}, \\ \tilde{b}(z) &= \sup\left\{ b \geqslant 0 : \mathbb{E}_{\mathbb{P}}\left[H(z) \cdot \mathbf{1}_{\{X_T > b)\}} \right] \leqslant (\tilde{C} - z)(1-\alpha) \right\}, \\ \gamma(z) &= 0. \end{aligned}$$

Denote

$$\zeta(b,z) = x_0 \tilde{\Phi}_+(b) - K(z)\tilde{\Phi}_-(b),$$

then

$$\mathbb{E}_{\mathbb{P}}\left[H(z) \cdot \mathbf{1}_{\{X_T > b\}} \right] = \begin{cases} \mathbb{E}_{\mathbb{P}}[H(z)], \text{ for } b < K(z), \\ \zeta(b,z), \qquad \text{for } b \geqslant K(z), \end{cases}$$

and

$$\zeta(K(z), z) = \mathbb{E}_{\mathbb{P}}[H(z)], \qquad \zeta(+\infty, z) = 0, \qquad \frac{\partial}{\partial b}\zeta(b,z) \leqslant 0.$$

The supremum is attained on the set $b \geqslant K(z)$, and $\tilde{b}(z)$ is a unique solution for the system

$$\begin{cases} x_0 \tilde{\Phi}_+(b) - K(z)\tilde{\Phi}_-(b) = (\tilde{C} - z)(1-\alpha), \\ b \geqslant K(z). \end{cases} \tag{3.9}$$

The optimal strategy $(\hat{V}_0, \hat{\xi})$ would be a perfect hedge for the contingent claim $H(\hat{z})^+ \mathbf{1}_{\left\{X_T < \tilde{b}(\hat{z})\right\}}$, where $\tilde{b}(z)$ is a solution for (3.9) and $\hat{z}$ is a point of minimum of function

$$d(z) = x_0 \Lambda_+(K(z), \tilde{b}(z)) - K(z)\Lambda_-(K(z), \tilde{b}(z))$$

on interval $z \in (-\infty, \tilde{C})$.

Now we apply the above results to the Black-Scholes model with the same parameters as in Section 3.2 ($\sigma = 0.3$, $\mu = 0.09$, $r = 0.05$, call option with strike price of $K = 110$, time to maturity $T = 0.25$, initial price $S_0 = 100$). Figure 2 shows the minimum amount of the initial wealth to be invested in the hedging strategy so that the resulting $\text{CVaR}_{0.975}$ does not exceed a specified threshold.

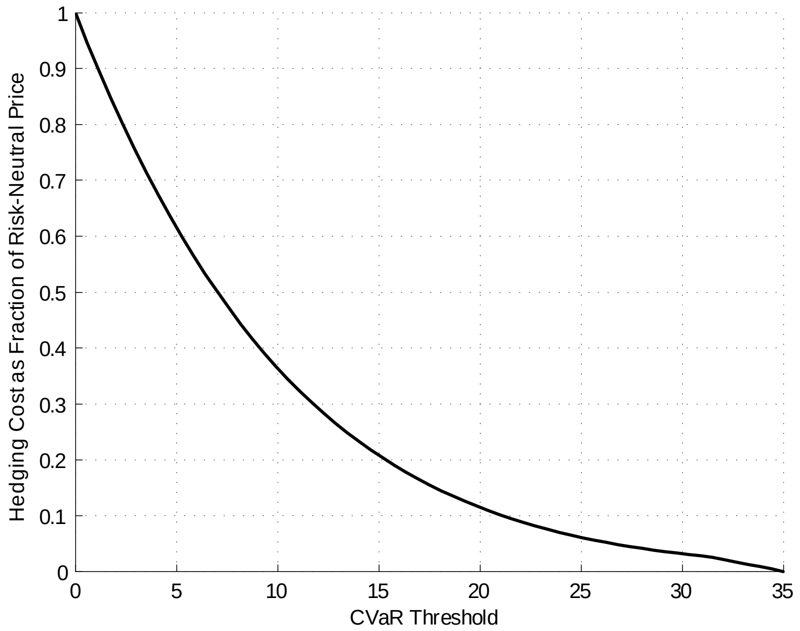

Figure 2. Minimizing the amount of the initial wealth for varying levels of CVaR threshold.

4 CVaR-Hedging of Equity-Linked Insurance Contracts

In this section we will use the quantile methodology proposed in papers by Melnikov and Skornyakova (2005) and Melnikov and Romaniuk (2006) to construct CVaR-optimal hedges of an embedded call option in an equity-linked life insurance contract.

In addition to the "financial" probability space $(\Omega, \mathcal{F}, \mathbb{P})$ introduced earlier, let us consider the "actuarial" probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. Let a random variable $T(x)$ denote the remaining lifetime of a person aged x, and let ${}_Tp_x = \tilde{\mathbb{P}}[T(x) > T]$ be a survival probability for the next T years of the insured. We assume that $T(x)$ does not depend on the evolution of financial market, so we can treat $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ as independent.

Under an equity-linked pure endowment contract, the insurance company is obliged to pay the benefit in the amount of $\bar{H}$ (an $\mathcal{F}_T$-measurable random variable) to the insured provided the insured is alive at time T. Essentially, the benefit is linked to the evolution of financial market, hence an insurance contract of this kind poses two independent kinds of risk to the insurance company: mortality risk and market risk.

According to the option pricing theory, the optimal price is traditionally calculated as an expected present value of cash flows under the risk-neutral probability. However, the insurance part of the contract doesn't need to be risk-adjusted since the mortality risk is essentially unsystematic. Denote the discounted benefit by $H = \bar{H}e^{-rT}$, then the price of the contract (known as "the Brennan-Schwartz price", see Brennan & Schwartz, 1976) shall be equal to

$$ {}_TU_x = \mathbb{E}_{\tilde{\mathbb{P}}}\left[\mathbb{E}_{\mathbb{P}^*}\left[H \cdot \mathbf{1}_{\{T(x)>T\}}\right]\right] = {}_Tp_x \cdot \mathbb{E}_{\mathbb{P}^*}\left[H\right]. $$

The problem of the insurance company is to mitigate the financial part of risk and hedge $\bar{H}$ in the financial market. However,

$$ {}_TU_x < \mathbb{E}_{\mathbb{P}^*}\left[H\right], $$

in other words, the insurance company is not able to hedge the benefit perfectly; instead, the benefit may be hedged partially.

For a fixed client age x, denote the maximum amount of capital that is going into partial hedging of $\bar{H}$ by $\tilde{V}_0 = {}_Tp_x \cdot \mathbb{E}_{\mathbb{P}^*}\left[H\right]$; we can now use the results of Theorem 2.4 to derive CVaR-optimal hedging strategy. Along with providing a way of hedging, this may be viewed as a possible way of estimating financial exposure of contracts for given values of age. Note that by applying Theorem

2.7 we can also address the dual problem: given the financial claim and a fixed CVaR threshold, we can find the target survival probability (and hence the target age) for the contract.

In the following example we investigate a pure endowment contract with a fixed guarantee which makes payment $\bar{H}$ at maturity provided the insured is alive:

$$\bar{H} = \max\{S_T, kS_0\},$$

where S_t is the stock price process and k is a fixed percentage value. Since

$$\max\{S_T, kS_0\} = kS_0 + (S_T - kS_0)^+,$$

it is sufficient for our purposes to only consider the embedded call $(S_T - K)^+$, where $K = kS_0$.

Let the financial part of our model follow the Black-Scholes with parameters $\sigma = 0.3$, $\mu = 0.09$, $r = 0.05$ and let the embedded call option have the strike price of $K = 110$; time horizon T will vary in this example. Let the initial price of the stock be equal to $S_0 = 100$. As for the insurance part, we shall use survival probabilities listed in mortality table UP94 @2015 from McGill, Brown, Haley, and Schieber (2004) (Uninsured Pensioner Mortality 1994 Table Projected to the Year 2015). Our objective here is to construct hedging strategies that minimize $\mathrm{CVaR}_{0.975}$ for varying values of client age and time horizon. Please note that since we are dealing with the Black-Scholes, we can refer to Section 3.2 for the calculation of optimal CVaR for a given amount of initial wealth. The numeric results are presented in Figure 3.

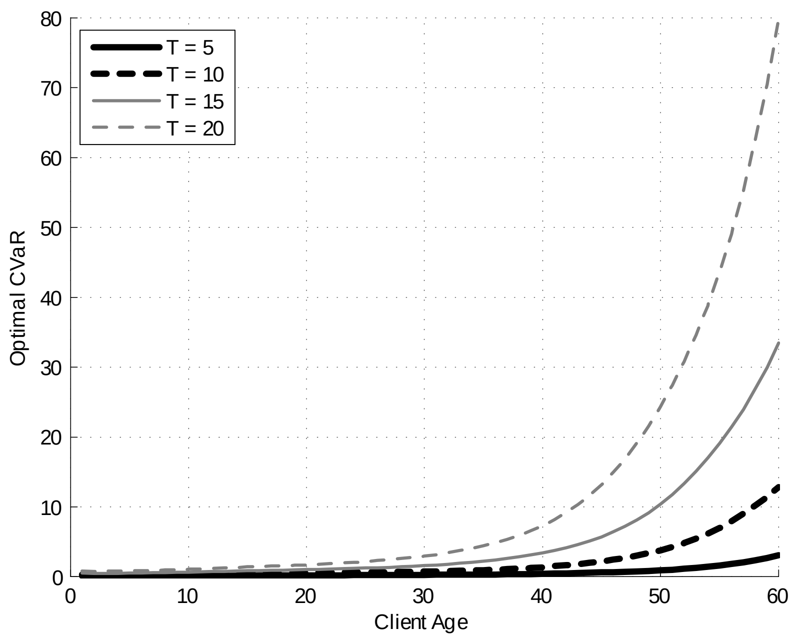

Figure 3. CVaR vs. age for unit-linked contracts of different maturity.

Now consider the dual problem: for a fixed CVaR threshold $\tilde{C}$, specify the optimal client age for the equity-linked life insurance contract. Assuming the Black-Scholes setting, we can employ the results of Section 3.3 to derive the optimal survival probability. Then it's just the matter of using the corresponding life table to find the optimal client age. (Note: depending on the life table, the client age may not be uniquely defined by the survival probability; in our example, we pick the highest possible value). To illustrate the above, we use model parameters from the previous example and survival probabilities from mortality table UP94 @2015, McGill et al. (2004); the results are presented in Figure 4.

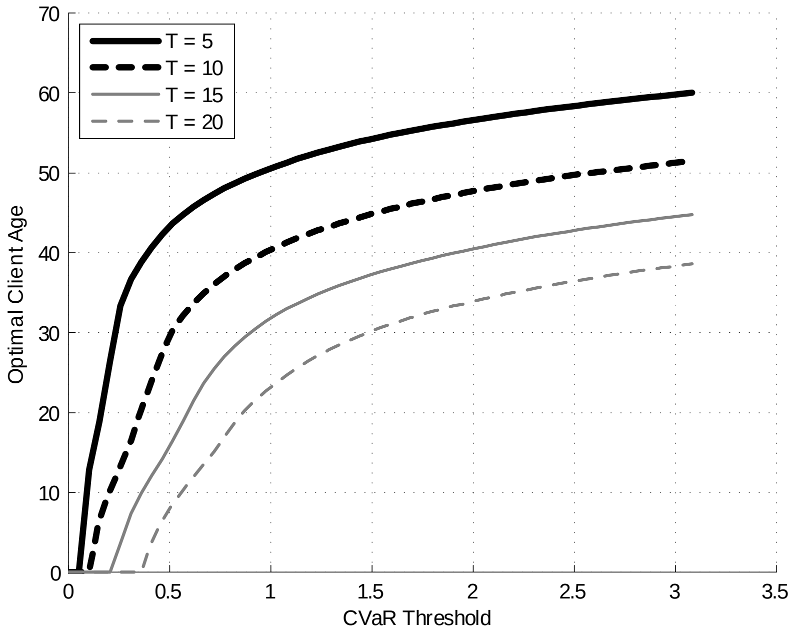

Figure 4. Age vs. CVaR for unit-linked contracts of different maturity.

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Highlights

We construct CVaR-optimal hedges subject to constraints on the initial wealth.
We also discuss how to minimize hedging costs subject to a CVaR constraint.
The approach is illustrated by deriving closed-form solutions in the Black-Scholes.
A practical application: CVaR-hedging a unit-linked life insurance contract.