CHAPTER 5

Plastic Equilibrium in Soils

ARTICLE 26 FUNDAMENTAL ASSUMPTIONS

This chapter deals with the earth pressure against lateral supports such as retaining walls or the bracing in open cuts, with the resistance of the earth against lateral displacement, with the bearing capacity of footings, and with the stability of slopes. Problems of this type require comparing the magnitudes of two sets of forces: those that tend to resist a failure and those that tend to produce it. Such an investigation is called a *stability calculation.* To make a stability calculation, the position of the potential surface of sliding must be determined and the resistance against sliding along this surface must be computed or estimated.

In reality both the forces tending to resist failure and those tending to produce it involve inherent uncertainties and, consequently, so does the *factor of safety,* defined as the quotient of these two quantities. The probabilistic nature of both the reistances and the driving forces has been the subject of considerable study and is recognized in building codes applicable to structures and their foundations; it will be discussed in Part III. In this chapter the principles of stability calculations are presented as if the pertinent quantities can be evaluated deterministically.

The sliding resistance s per unit of area depends not only on the type of soil but also on the effective normal stress $\sigma - u$ on the surface of sliding and a number of other factors. These were discussed in Articles 18 through 20.

For mathematical treatment of stability problems, simplified expressions for shear strength are used. For dry or saturated granular soils, for saturated normally consolidated clays, and for fully softened stiff clays:

$$s = (\sigma - u) \tan \phi' \tag{26.1}$$

where ϕ' is a friction angle independent of the effective normal stress $\sigma - u$ at failure and is constant throughout the soil. For saturated overconsolidated or aged clays,

$$s = c' + (\sigma - u) \tan \phi' \tag{26.2}$$

where c' and ϕ' are a cohesion intercept and a friction angle, respectively; both are independent of the effective normal stress at failure and are constant throughout the soil mass. For unsaturated cohesive soils,

$$s = c + \sigma \tan \phi' \tag{26.3}$$

where $c = c' + b\, u_s^a$ (Eq. 19.9), and c' and ϕ' are a cohesion intercept and a friction angle, respectively. The parameters a and b as well as suction u_s are assumed not to vary throughout the soil mass; constant values of c and ϕ', independent of total normal stress at failure σ, are used in the analysis. In reality the failure envelopes of dense granular soils and overconsolidated clays are curved, and c' and u_s may vary with depth. Thus the selection of appropriate values of ϕ', c' and c for a particular problem requires experience and judgment (Article 19).

The investigation of failure based on Eqs. 26.1 or 26.2 is called *effective stress stability analysis* (ESSA). The ESSA terminology is also applicable to Eq. 26.3, because in unsaturated soils with air voids that are connected to the atmosphere and with constant suction, the frictional shearing resistance determined by ϕ' is directly proportional to σ. When ESSA is applied to drained failures, the porewater pressure in Eqs. 26.1 and 26.2 is determined from the ground water level, or from a flow net if steady seepage exists. When ESSA is used for mathematical analysis of undrained failures, the porewater pressure term must also include porewater pressures resulting from loading or unloading and associated shearing deformations (Article 15.5). Considerable care must be taken in selecting c' and ϕ' for undrained failure of contractive soils (Article 20.1).

More often, the shear strength for constant volume undrained failures is defined as

$$s = s_u \tag{26.4}$$

where s_u is an average mobilized undrained shear strength assumed to have the same value at all depths and all directions. Equation 26.4 is most suitable for undrained stability analyses of saturated soft clays and silts and for loose sands, in which the undrained shear strength s_u is independent of the total normal stress. An investigation of failure based on Eq. 26.4 is called an *undrained strength stability analysis* (USSA). Because the undrained shear strength may vary with depth and is highly dependent on mode of shear and time to failure, the selection of the appropriate value of the average mobilized undrained shear strength for a particular problem requires experience and judgment (Article 20). When the undrained shear strength varies with depth in an irregular but distinct manner, the soil should be considered to consist of layers having different values of average mobilized undrained shear strength. In rare instances, such as in an earth dam constructed from a homogeneous cohesive soil, the undrained shear strength may increase linearly with depth or with the consolidation pressure σ'_{vo}. The relationship between undrained shear strength and consolidation pressure may be included in a mathematical analysis as a linear relationship between undrained shear strength and effective normal stress before loading or unloading; i.e., before the associated shear deformation.

Each of the stability problems will be solved first for a dry ($u = 0$) cohesionless sand to which Eq. 26.1 is applicable and then for a cohesive material to which Eqs. 26.2 or 26.3 applies. After the ability to solve problems on the basis of these two equations is achieved, similar problems dealing with partly or completely submerged sand or with saturated clay under undrained conditions can readily be solved.

In a partly submerged mass of sand in which the water is at rest, the neutral stress u at any depth z below the water table is

$$u = \gamma_w z$$

This stress reduces the effective unit weight of that part of the sand below water level from γ to the submerged unit weight γ' (Eq. 15.6). Hence a stability calculation dealing with a partly submerged sand can be made on the assumption that the sand is dry, provided that the unit weight γ of the soil below water level is replaced by γ'. The pressure exerted by a partly submerged mass of sand against a lateral support is equal to the sum of the pressure of the sand, computed on the basis just mentioned, and the full water pressure. However, if the water percolates through the voids of the soil instead of being stagnant, this procedure is not applicable because the seepage pressure of the percolating water must be taken into account. Problems dealing with seepage pressure are discussed in Articles 35 and 36.

Theoretical expressions derived on the basis of Eq. 26.2 or Eq. 26.3 can be applied to stability analyses in which the shear strength is expressed by Eq. 26.4. The equation for the undrained shear strength of saturated soils in terms of *total stress* is

$$s = c + \sigma \tan \phi \tag{26.5}$$

where $c = s_u$ and $\phi = 0$. Because the mathematical forms of Eqs. 26.2, 26.3, and 26.5 are identical, they lead to the same mathematical expressions for solving stability problems. Thus, by appropriate substitutions for the intercept and angle, mathematical expressions for the undrained stability of saturated soils are obtained in terms of s_u.

The condition for failure represented by Eq. 26.2 corresponds to Mohr's rupture diagram in which the failure envelope is a straight line (Fig. 26.1). Consequently, a definite relation exists at failure between the major and minor principal stresses σ'_1 and σ'_3, respectively. By geometry

$$\sigma'_1 + d = OA + AB = OA(1 + \sin \phi')$$

$$\sigma'_3 + d = OA - AB = OA(1 - \sin \phi')$$

whence

$$\sigma'_1 = \sigma'_3 \frac{1 + \sin \phi'}{1 - \sin \phi'} + d\left(\frac{1 + \sin \phi'}{1 - \sin \phi'} - 1\right)$$

But

$$d = c' \frac{\cos \phi'}{\sin \phi'} = c' \frac{\sqrt{1 - \sin^2 \phi'}}{\sin \phi'}$$

Therefore

$$\sigma'_1 = \sigma'_3 \frac{1 + \sin \phi'}{1 - \sin \phi'} + 2c' \sqrt{\frac{1 + \sin \phi'}{1 - \sin \phi'}}$$

$$\sigma'_1 = \sigma'_3 \tan^2\left(45° + \frac{\phi'}{2}\right) + 2c' \tan\left(45° + \frac{\phi'}{2}\right)$$

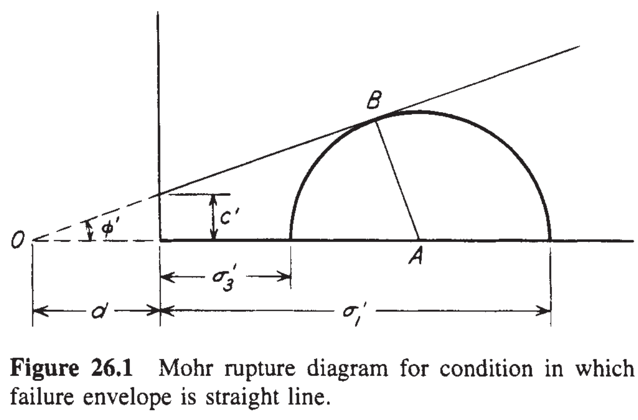

Figure 26.1 Mohr rupture diagram for condition in which failure envelope is straight line.

Or, if

$$N_{\phi'} = \tan^2\left(45° + \frac{\phi'}{2}\right) \tag{26.6}$$

$$\sigma_1' = \sigma_3' N_{\phi'} + 2c'\sqrt{N_{\phi'}} \tag{26.7}$$

The quantity $N_{\phi'}$ is known as the *flow value*. If $c' = 0$,

$$\sigma_1' = \sigma_3' N_{\phi'} \tag{26.8}$$

and if the condition for failure is represented by Eq. 26.4 ($\phi = 0$, $N_\phi = 1$)

$$\sigma_1 = \sigma_3 + 2s_u \tag{26.9}$$

ARTICLE 27 STATES OF PLASTIC EQUILIBRIUM

27.1 Fundamental Concepts

A body of soil is in a *state of plastic equilibrium* if every part of it is on the verge of failure. Rankine (1857) investigated the stress conditions corresponding to those states of plastic equilibrium that can be developed simultaneously throughout a semi-infinite mass of soil acted on by no force other than gravity. States of plastic equilibrium identical with those which Rankine considered are referred to as *Rankine states of plastic equilibrium.* A discussion of the Rankine states in a semi-infinite mass is primarily an introduction to the more complicated states of plastic equilibrium encountered in connection with practical problems.

The Rankine states are illustrated by Fig. 27.1. In this figure, AB represents the horizontal surface of a semi-infinite mass of cohesionless dry sand with a unit weight γ, and E represents an element of the sand with a depth z and a cross-sectional area equal to unity. Because the element is symmetrical with reference to a vertical plane, the effective normal stress on the base

$$\sigma_v' = \gamma z \tag{27.1}$$

is a principal stress. As a consequence, the effective normal stresses σ_h' on the vertical sides of the element at depth z are also principal stresses.

According to Eq. 26.8, the ratio between the major and minor principal stresses in a cohesionless material cannot exceed the value

$$\frac{\sigma_1'}{\sigma_3'} = N_{\phi'} = \tan^2\left(45° + \frac{\phi'}{2}\right)$$

Because the vertical principal stress σ_v', in the mass of sand shown in Fig. 27.1*a*, can be either the major or the minor principal stress, the ratio $K = \sigma_h'/\sigma_v'$ can assume any value between the limits,

$$K_A = \frac{\sigma_h'}{\sigma_v'} = \frac{1}{N_{\phi'}} = \tan^2\left(45° - \frac{\phi'}{2}\right) \tag{27.2}$$

and

$$K_P = \frac{\sigma_h'}{\sigma_v'} = N_{\phi'} = \tan^2\left(45° + \frac{\phi'}{2}\right) \tag{27.3}$$

After a mass of sand has been deposited by either a natural or an artificial process, K has a value K_0 intermediate between K_A and K_P, and

$$\sigma_h' = K_0\sigma_v' \tag{27.4}$$

wherein K_0 is an empirical coefficient known as the *coefficient of earth pressure at rest.* Its value depends on the relative density of the sand, the process by which the deposit was formed, and its subsequent stress history (Article 16.5).

To change the value of K for a mass of sand from K_0 to some other value, it is necessary to give the entire mass an opportunity either to stretch or to be compressed in a horizontal direction. Because the weight of sand above any horizontal section remains unchanged, the effective vertical pressure σ_v', is unaltered. The horizontal pressure $\sigma_h' = K\sigma_v'$, however, decreases if the mass stretches and increases if it compresses.

As the mass stretches, any two vertical sections such as ab and cd move apart, and the value of K decreases until it becomes equal to K_A (Eq. 27.2). The sand is then in what is known as the *active Rankine state.* In this state the intensity of the effective horizontal pressure at any depth z is equal to

$$\sigma_h' = K_A\sigma_v' = K_A\gamma z = \gamma z\frac{1}{N_{\phi'}} \tag{27.5}$$

in which K_A is called the *coefficient of active earth pressure.* The distribution of pressure over the sides and base of an element such as E is shown in Fig. 27.1*b*. Further stretching of the mass has no effect on σ_h' (Eq. 27.5), but sliding occurs along two sets of plane surfaces as indicated on the right-hand side of Fig. 27.1*a*. According to Eq. 17.5, such surfaces of sliding intersect the direction of the minor principal stress at the angle $45° + \phi'/2$. Because the minor principal stresses in the active Rankine state are horizontal, the shear planes rise at an angle of $45° + \phi'/2$ with the horizontal. The pattern formed by the traces of the shear planes on a vertical section parallel to the direction of stretching is known as the *shear pattern.*

A horizontal compression of the entire mass of sand causes ab to move toward cd, as shown in Fig. 27.1*c*. As a consequence, the ratio $K = \sigma_h'/\sigma_v'$ increases. As soon as K becomes equal to K_P (Eq. 27.3) the sand is

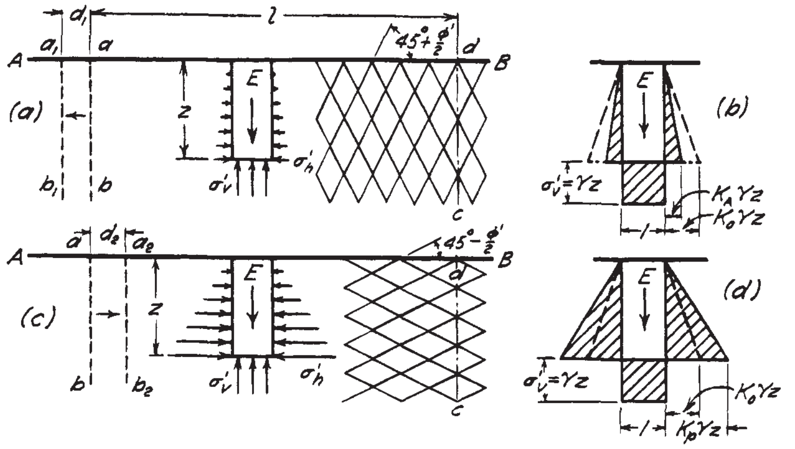

Figure 27.1 (*a, b*) Diagrams illustrating active Rankine state in semi-infinite mass of sand; (*c, d*) corresponding diagrams for passive Rankine state.

said to be in the *passive Rankine state.* At any depth z the horizontal pressure is

$$\sigma'_h = K_P \sigma'_v = K_P \gamma z = \gamma z N_{\phi'} \qquad (27.6)$$

in which K_P is the *coefficient of passive earth pressure.* Because the minor principal stress in the passive Rankine state is vertical, the surfaces of sliding rise at an angle of $45° - \phi'/2$ with the horizontal, as shown in Fig. 27.1*c*.

The active and the passive Rankine states constitute the two limiting states for the equilibrium of the sand. Every intermediate state, including the state of rest, is referred to as a *state of elastic equilibrium.*

27.2 Local States of Plastic Equilibrium

The Rankine states illustrated by Fig. 27.1 were produced by uniformly stretching or compressing every part of a semi-infinite mass of sand. They are known as *general states of plastic equilibrium.* However, in a stratum of real sand, no general state of equilibrium can be produced except by a geologic process such as the horizontal compression by tectonic forces of the entire rock foundation of the sand strata. Local events, such as the yielding of a retaining wall, cannot produce a radical change in the state of stress in the sand except in the immediate vicinity of the source of the disturbance. The rest of the sand remains in a state of elastic equilibrium.

Local states of plastic equilibrium can be produced by very different processes of deformation. The resulting states of stress in the plastic zone and the shape of the zone itself depend to a large extent on the type of deformation and on the degree of roughness of the surface of contact between the soil and its support. These factors constitute the *deformation* and the *boundary conditions.* The practical consequences of these conditions are illustrated by Figs. 27.2 and 27.3.

Figure 27.2*a* is a vertical section through a prismatic box having a length l equal to the distance between the vertical sections *ab* and *cd* in Fig. 27.1. If dry sand is deposited in the box by the same process that was responsible for the formation of the semi-infinite mass represented in Fig. 27.1, the states of stress in both masses are identical. They represent states of elastic equilibrium.

When the state of the semi-infinite mass of sand (Fig. 27.1*a*) was changed from that of rest to the active Rankine state, the vertical section *ab* moved through the distance d_1. To change the state of the entire mass of sand contained in the box (Fig. 27.2*a*) into the active Rankine state, the wall *ab* must be moved through the same distance. This constitutes the deformation condition. While the wall *ab* (Fig. 27.2*a*) moves out, the height of the mass of sand decreases, and its length increases. These movements involve displacements between the sand and all the surfaces of the box which it contacts. If the contact surfaces are rough, shearing stresses will develop along vertical and horizontal planes. Because the shearing stresses on these planes are zero in the active Rankine state, this state cannot materialize unless the sides and bottom of the box are perfectly smooth. This requirement constitutes the boundary condition for the transition of the sand in the box to the active Rankine state. If this condition is satisfied, the sand passes into an active Rankine state as soon as the wall *ab* reaches the position a_1b_1. At this stage, the unit stretch of the soil is d_1/l. Any further movement of the wall causes slippage along the two sets of surfaces of sliding indicated by dash lines in Fig. 27.2*a,* but the stress conditions remain unchanged.

If the wall *ab* is perfectly smooth but the bottom of the box is rough, the sand located between the wall *ab* and the potential surface of sliding *be* is free to deform in exactly the same manner as it does in a box with a smooth bottom, but the state of stress in the balance of the sand cannot change materially because the friction along the bottom prevents the required deformation. Hence, an outward movement of the wall *ab* produces

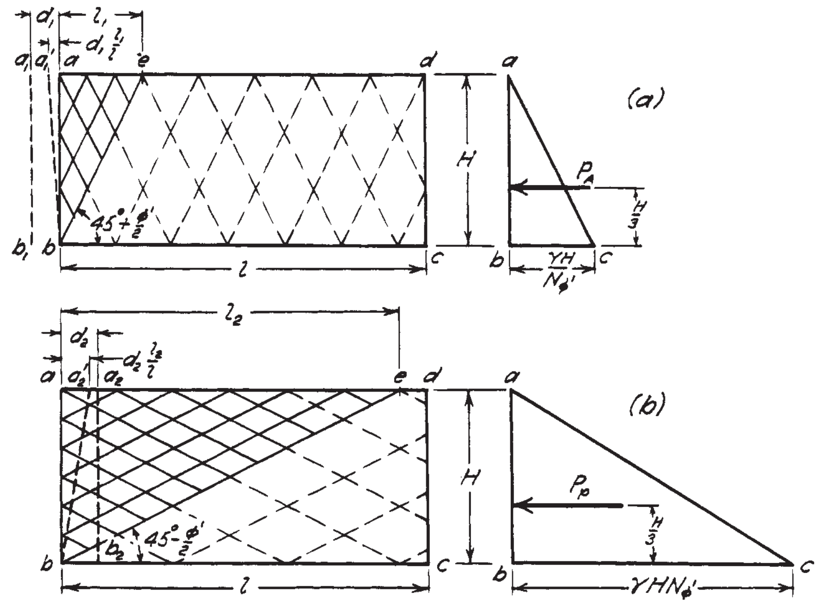

Figure 27.2 (*a*) Diagrams illustrating local active Rankine state in sand contained in rectangular box; (*b*) corresponding diagrams for local passive Rankine state.

an active Rankine state within only the wedge-shaped zone *abe*. Because the width of the wedge increases from zero at the bottom to l_1 at the top, the unit stretch d_1/l required to establish the active Rankine state in the wedge is attained as soon as the left-hand boundary of the wedge moves from *ab* to $a_1'b$ (Fig. 27.2*a*). This is the deformation condition for the development of an active Rankine state within the wedge. As soon as the wall *ab* passes beyond this position, the wedge slides downward and outward along a plane surface of sliding *be* which rises at an angle of $45° + \phi'/2$ with the horizontal.

If the wall *ab* is pushed toward the sand, and if both the walls and the bottom of the box are perfectly smooth, the entire mass of sand is transformed into the passive Rankine state (Fig. 27.2*b*) as soon as the wall moves beyond a distance d_2 from its original position. The planes of sliding rise at an angle of $45° - \phi'/2$ with the horizontal. If the wall *ab* is perfectly smooth but the bottom of the box is rough, the passive Rankine state develops only within the wedge-shaped zone *abe*. The transition from the elastic to the plastic state does not occur until *ab* moves into or beyond the position $a_2'b$.

If the end of the box is free to move outward at the bottom but is restrained at the top, as indicated in Fig. 27.3, the sand fails by shear along some surface of sliding as soon as the tilt becomes perceptible, because the deformations compatible with an elastic state of equilibrium are very small. However, even at the state of failure, the sand between the wall and the surface of sliding does not pass into the active Rankine state because the upper part of the wall cannot move, and, as a consequence, the deformation condition for the active Rankine state within the sliding wedge is not satisfied.

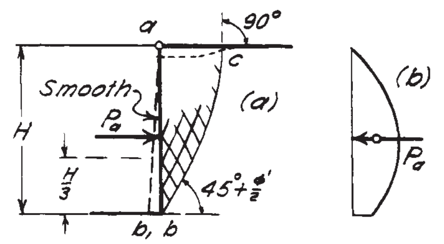

Figure 27.3 Failure of sand behind smooth vertical wall when deformation condition for active Rankine state is not satisfied. (*a*) Section through back of wall; (*b*) stress against back of wall.

Theoretical and experimental investigations regarding the type of failure caused by a tilt of the lateral support about its upper edge have led to the conclusion that the surface of sliding starts at *b* (Fig. 27.3*a*) at an angle of $45° + \phi'/2$ with the horizontal and that it becomes steeper until it intersects the ground surface at a right angle. The upper part of the sliding wedge remains in a state of elastic equilibrium until the lower part of the wedge has passed completely into a state of plastic equilibrium. The distribution of pressure against the lateral support is roughly parabolic (Fig. 27.3*b*) instead of triangular (Fig. 27.1*b*).

Similar investigations regarding the effect of pushing the bottom of the support toward the soil (Fig. 27.4*a*) have shown that the surface of sliding rises from *b* at an angle $45° - \phi'/2$ with the horizontal and that it also

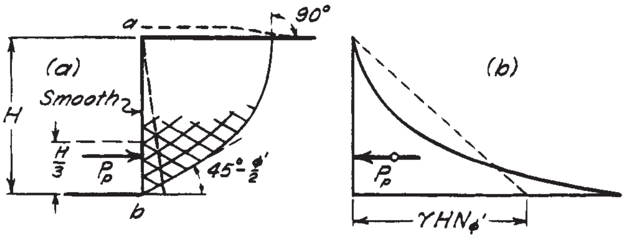

Figure 27.4 Failure of sand behind smooth vertical wall when deformation condition for passive Rankine state is not satisfied. (*a*) Section through back of wall; (*b*) stress on back of wall.

intersects the ground surface at a right angle. The corresponding distribution of pressure is shown in Fig. 27.4*b*.

Selected Reading

A general discussion of the application of the theory of plasticity to states of limiting equilibrium, including problems of earth pressure, stability of slopes, and bearing capacity, is contained in Sokolovski, V. V. (1960). *Statics of Soil Media.* Translated from the Russian by D. H. Jones and A. N. Schofield, London, Butterworths, 237 pp. Mathematical methods for solving problems with mixed boundary conditions are developed in Hansen, B. (1965). *A Theory of Plasticity for Ideal Frictionless Materials,* Copenhagen, Teknisk Forlag, 471 pp.

Chen, W.F. (1975). *Limit analysis and soil plasticity, Developments in Geotechnical Engineering,* **7,** Elsevier, Amsterdam, 638 pp.

ARTICLE 28 RANKINE'S EARTH-PRESSURE THEORY

28.1 Earth Pressure against Retaining Walls

Retaining walls serve the same function as the vertical sides of the box shown in Fig. 27.2. The soil adjoining the wall is known as the *backfill.* It is always deposited after the wall is built. While the backfill is being placed, the wall yields somewhat under the pressure. The ultimate value of the pressure depends not only on the nature of the soil and the height of the wall but also on the amount of yield. If the position of the wall is fixed, the earth pressure is likely to retain forever a value close to the earth pressure at rest (Article 27). However, as soon as a wall yields far enough, it automatically satisfies the deformation condition for the transition of the adjoining mass of soil from the state of rest into an active state of plastic equilibrium. Hence, the factor of safety of a retaining wall capable of yielding must be adequate with respect to the active earth pressure, but does not need to be investigated for greater values of earth pressure.

Although the back of every real retaining wall is rough, approximate values of the earth pressure can be obtained on the assumption that it is smooth. In the following paragraphs, this assumption is made. Methods for obtaining more accurate values will be described in subsequent articles.

28.2 Active Earth Pressure of Cohesionless Soil against Smooth Vertical Walls

If the surface of a sand backfill is horizontal, and if the back of the retaining wall is vertical and perfectly smooth, the magnitude and the distribution of pressure against the back of the wall are identical with those of the active pressure against the fictitious plane *ab* in Fig. 27.1*a.* Therefore, the earth pressure can be computed on the basis of the equations already derived. In reality, there are no perfectly smooth surfaces. However, the equations based on this assumption are so simple that they are quite commonly used for evaluating the earth pressure against real retaining walls and other structures acted on by earth pressure. It is shown subsequently that the roughness of the back of a wall commonly reduces the active and increases the passive earth pressure. Hence, as a rule, the error associated with the assumption is on the safe side.

Furthermore, in one case of great practical importance, the assumption of a smooth vertical wall is almost strictly correct. This case is illustrated by Fig. 28.1, which represents a cantilever wall. If such a wall yields under the influence of the earth pressure, the sand fails by shear along two planes rising from the heel of the wall at angles of $45° + \phi'/2$ with the horizontal. Within the wedge-shaped zone located between these two planes, the sand is in the active Rankine state, and no shearing stresses act along the vertical plane *ab* through the heel. Hence, the earth pressure against this plane is identical with that against a smooth vertical wall.

If the sand backfill is perfectly dry, the active pressure against a smooth vertical wall at any depth z is

$$\sigma'_h = \gamma z \frac{1}{N_{\phi'}} \tag{27.5}$$

It increases in simple proportion to the depth, as indicated

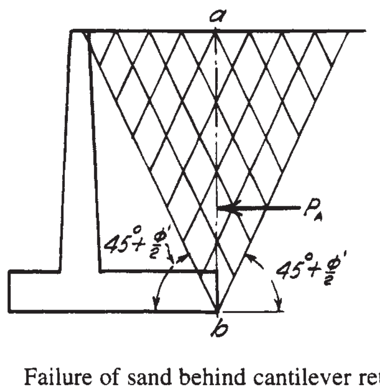

Figure 28.1 Failure of sand behind cantilever retaining wall; deformation condition for active Rankine state is almost satisfied.

by the pressure triangle *abc* (Fig. 27.2*a*). The resultant pressure or force against the wall is

$$P_A = \int_0^H \sigma_h' \, dz = \frac{1}{2} \gamma H^2 \frac{1}{N_{\phi'}} \qquad (28.1)$$

The point of application of P_A is located at a height $H/3$ above *b*.

If the wall is pushed into the position $a_2'b$ in Fig. 27.2*b* the pressure σ_h' against the wall assumes a value corresponding to the passive Rankine state,

$$\sigma_h' = \gamma z N_{\phi'} \qquad (27.6)$$

and the resultant pressure against the wall becomes equal to

$$P_P = \int_0^H \sigma_h' \, dz = \frac{1}{2} \gamma H^2 N_{\phi'} \qquad (28.2)$$

28.3 Active Earth Pressure of Partly Submerged Sand Supporting a Uniform Surcharge

In Fig. 28.2*a* the line *ab* represents the smooth vertical back of a wall with height *H*. The effective unit weight of the sand when dry is γ and when submerged is γ' (see Article 15); the unit weight of water is γ_w. The surface of the horizontal backfill carries a uniformly distributed surcharge *q* per unit of area. Within the backfill the water table is located at depth H_1 below the crest of the wall. The angle of internal friction of both the dry and submerged sand is assumed to be ϕ'.

As the wall yields from position *ab* into position $a_1'b$, the pressure against its back decreases from the value of the earth pressure at rest to that of the active Rankine pressure. In Article 26, it was shown that the entire effect of the porewater pressure on the effective stresses in the sand can be taken into account by assigning to the submerged part of the sand the submerged unit weight γ' (Eq. 15.6). Within the depth H_1 the pressure on the wall due to the weight of the adjoining sand is represented by the triangle *ace* in Fig. 28.2*b*. At any depth z' below the water table the effective vertical pressure on a horizontal section through the sand is

$$\sigma_v' = H_1\gamma + z'\gamma'$$

For the corresponding horizontal active Rankine pressure we obtain by means of Eq. 27.5

$$\sigma_h' = \frac{\sigma_v'}{N_{\phi'}} = (H_1\gamma + z'\gamma') \frac{1}{N_{\phi'}} \qquad (28.3)$$

The resultant effective horizontal pressure below the water level is represented by the area *bced* in Fig. 28.2*b*. To this force must be added the resultant water pressure,

$$P_w = {}^1\!/_2 \, \gamma_w H_2^2 \qquad (28.4)$$

which acts against the lower part *cb* of the wall. In Fig. 28.2*b*, the water pressure is represented by the triangle *def*.

If the fill carries a uniformly distributed surcharge *q* per unit of area, the effective vertical stress σ_v' increases at any depth by *q*, and the corresponding horizontal active Rankine pressure increases by

$$\Delta\sigma_h' = \frac{q}{N_{\phi'}} \qquad (28.5)$$

In Fig. 28.2*b* the pressure produced by the surcharge *q* is represented by the area *aefihg*.

28.4 Active Earth Pressure of Cohesive Soils against Smooth Vertical Surfaces

In Fig. 28.3*a* the line *ab* represents the smooth vertical back of a wall in contact with an unsaturated cohesive soil having a unit weight γ located above the water table. The shearing resistance of the soil is defined by Eq. 26.3

$$s = c + \sigma \tan \phi'$$

The relation between the extreme values of the principal

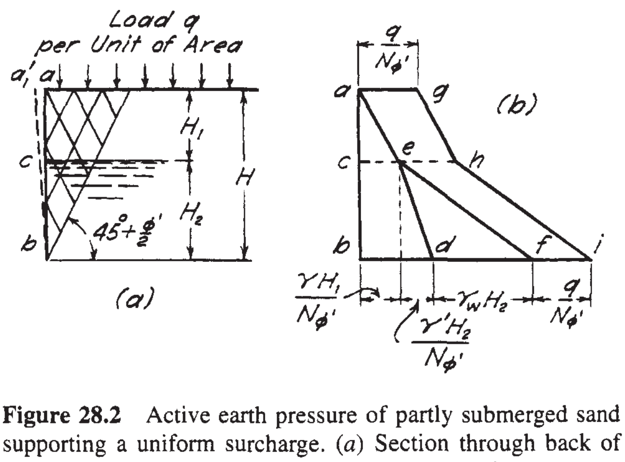

Figure 28.2 Active earth pressure of partly submerged sand supporting a uniform surcharge. (*a*) Section through back of supporting structure; (*b*) pressure against back of structure.

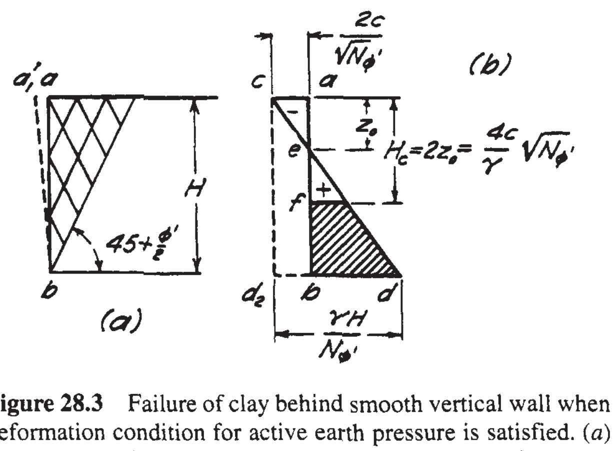

Figure 28.3 Failure of clay behind smooth vertical wall when deformation condition for active earth pressure is satisfied. (*a*) Section through back of wall; (*b*) pressure against back of wall.

stresses in such soils is determined by the expression,

$$\sigma_1 = \sigma_3 N_{\phi'} + 2c\sqrt{N_{\phi'}} \tag{26.7}$$

wherein σ_1 and σ_3 are the major and minor total principal stresses, respectively, and

$$N_{\phi'} = \tan^2\left(45° + \frac{\phi'}{2}\right) \tag{26.6}$$

is the flow value. In Article 17.4 it is also shown that the surfaces of sliding intersect the direction of the minor principal stress at an angle $45° + \phi'/2$, regardless of the value of c (Eq. 26.2).

Because the back of the wall is smooth, the vertical principal stress at depth z below the horizontal surface of the backfill is $\sigma_v = \gamma z$. Before the support ab moves, it is acted on by the earth pressure at rest. In this state the horizontal stress σ_h is the minor principal stress. An outward movement of the support into or beyond the position $a_1'b$ reduces σ_h to the value corresponding to the active Rankine pressure. Substituting $\sigma_v = \sigma_1 = \gamma z$ and $\sigma_h = \sigma_3$ into Eq. 26.2, we obtain

$$\sigma_h = \gamma z \frac{1}{N_{\phi'}} - 2c\frac{1}{\sqrt{N_{\phi'}}} \tag{28.6}$$

This stress at any depth z is represented by the horizontal distance between the lines ab and cd in Fig. 28.3*b*. At depth,

$$z_0 = \frac{2c}{\gamma}\sqrt{N_{\phi'}} \tag{28.7}$$

the stress σ_h is equal to zero. At a depth less than z_0, the pressure against the wall is negative, provided that a crack does not open up between the wall and the uppermost part of the soil. The resultant horizontal earth pressure against the wall is

$$P_A = \int_0^H \sigma_h\,dz = \frac{1}{2}\gamma H^2\frac{1}{N_{\phi'}} - 2c\frac{H}{\sqrt{N_{\phi'}}} \tag{28.8}$$

If the wall has a height,

$$H = H_c = \frac{4c}{\gamma}\sqrt{N_{\phi'}} = 2z_0 \tag{28.9}$$

the resultant earth pressure P_A is equal to zero. Hence, if the height of a vertical bank is smaller than H_c, the bank should be able to stand without lateral support. However, the pressure against the wall increases from $-2c/\sqrt{N_{\phi'}}$ at the crest to $+2c/\sqrt{N_{\phi'}}$ at depth H_c, whereas on the vertical face of an unsupported bank the normal stress is zero at every point. Because of this difference the greatest depth to which a cut can be excavated without lateral support of its vertical sides is slightly smaller than H_c (see Article 35).

If the cohesive soil above the water table is practically saturated so that it is justifiable to express the extreme values of the principal stresses by Eq. 26.9, then

$$P_A = \tfrac{1}{2}\gamma H^2 - 2s_u H \tag{28.10}$$

and

$$H_c = \frac{4s_u}{\gamma} \tag{28.11}$$

Because the soil does not necessarily adhere to the wall, it is generally assumed that the active earth pressure of cohesive soils against retaining walls is equal to the pressure represented in Fig. 28.3*b* by the triangular area *bde*, equal to area cdd_2 − area $cebd_2$. Therefore,

$$P_A = \frac{1}{2}\gamma H^2\frac{1}{N_{\phi'}} - 2cH\frac{1}{\sqrt{N_{\phi'}}} + \frac{2c^2}{\gamma} \tag{28.12}$$

In terms of Eq. 26.9,

$$P_A = \frac{1}{2}\gamma H^2 - 2s_u H + \frac{2s_u^2}{\gamma} \tag{28.13}$$

28.5 Passive Earth Pressure of Cohesive Soils in Contact with Smooth Vertical Surfaces

If the face ab of the wall or block that supports the soil and its uniform surcharge q is pushed toward the backfill as indicated in Fig. 28.4*a*, the horizontal principal stress σ_h increases and becomes greater than σ_v. As soon as ab arrives at or beyond the position $a_2'b$, which represents the deformation condition for the passive Rankine state, the stress conditions for failure (Eq. 26.2) are satisfied. Because σ_h represents the major principal stress, we may substitute $\sigma_h = \sigma_1$ and $\sigma_v = \sigma_3 = \gamma z + q$ into Eq. 26.2 and obtain

$$\sigma_h = \gamma z N_{\phi'} + 2c\sqrt{N_{\phi'}} + qN_{\phi'} \tag{28.14}$$

The stress σ_h can be resolved into two parts. One part

$$[\sigma_h]_I = \gamma z N_{\phi'}$$

increases like a hydrostatic pressure in simple proportion to depth. In Fig. 28.4*b* the stresses $[\sigma_h]_I$ are represented by the width of the triangle $c_1c_2d_2$ with the area

$$[P_P]_I = \tfrac{1}{2}\gamma H^2 N_{\phi'} \tag{28.15}$$

The point of application of $[P_P]_I$ is located at an elevation $H/3$ above b. The quantity $[P_P]_I$ represents the resultant passive earth pressure of a cohesionless material with an angle of internal friction ϕ' and a unit weight γ.

The second part of σ_h is

$$[\sigma_h]_{II} = 2c\sqrt{N_{\phi'}} + qN_{\phi'}$$

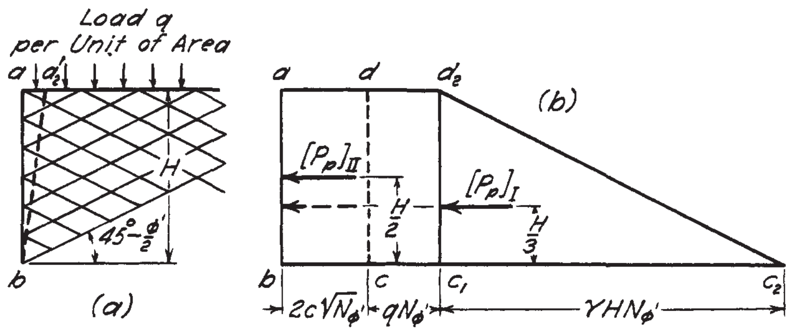

Figure 28.4 Failure of clay behind smooth vertical wall when deformation condition for passive earth pressure is satisfied. (*a*) Section through back of wall; (*b*) stress on back of wall.

This part is independent of the depth. It is represented by the width of the rectangle abc_1d_2 in Fig. 28.4*b*. The horizontal force is equal to the area of the rectangle. Hence,

$$[P_P]_{II} = H(2c\sqrt{N_{\phi'}} + qN_{\phi'}) \qquad (28.16)$$

The point of application of $[P_P]_{II}$ is at mid-height of the surface *ab*. Since Eq. 28.16 does not contain the unit weight γ, the value $[P_P]_{II}$ can be computed on the assumption that the backfill is weightless. From Eqs. 28.15 and 28.16, we find that the resultant passive earth pressure is

$$P_P = [P_P]_I + [P_P]_{II} = \tfrac{1}{2}\gamma H^2 N_{\phi'} + H(2c\sqrt{N_{\phi'}} + qN_{\phi'}) \qquad (28.17)$$

According to the preceding discussion, P_P can be computed by two independent operations. First, $[P_P]_I$ is computed on the assumption that the cohesion and the surcharge are zero ($c = 0, q = 0$). The point of application of $[P_P]_I$ is located at the lower third point of *H*. Second, $[P_P]_{II}$ is computed on the assumption that the unit weight of the backfill is zero ($\gamma = 0$). The point of application of $[P_P]_{II}$ is at the midpoint of *H*. In the following articles this simple procedure is used repeatedly for determining the point of application of the passive earth pressure of cohesive soils. The subdivision of P_P into the two parts $[P_P]_I$ and $[P_P]_{II}$ is strictly correct only when the back of the wall is vertical and perfectly smooth. For all other conditions, the procedure is approximate.

Problems

1. A wall with a smooth vertical back 3 m high retains a mass of dry cohesionless sand that has a horizontal surface. The sand weighs 18 kN/m^3 and has an angle of internal friction of 36°. What is the approximate resultant pressure against the wall, if the wall is prevented from yielding? If the wall can yield far enough to satisfy the deformation condition for the active Rankine state?

Ans. A unique relationship between K_0 and ϕ' for densified sands does not exist. Using the correlation between K_0 and ϕ' in Fig. 44.6 and assuming $K_{op} = 0.5$ leads to $K_0 = 0.65$; 53 kN/m; 21 kN/m.

2. The water level behind the wall described in problem 1 rises to an elevation 1 m below the crest. The submerged unit weight of the sand is 10 kN/m^3. If the deformation condition for the active Rankine state is satisfied, what is the resultant pressure that the earth and water exert against the wall? At what height above the base does the resultant of the earth and water pressures act?

Ans. 37 kN/m; 0.86 m.

3. What is the resultant lateral pressure against the yielding wall in problem 1, if the sand mass supports a uniformly distributed load of 20 kPa? At what height above the base of the wall is the center of pressure?

Ans. 37 kN/m; 1.2 m.

4. The space between two retaining walls with smooth backs is filled with sand weighing 18 kN/m^3. The foundations of the walls are interconnected by a reinforced concrete floor, and the crests of the walls by heavy steel tie rods. The walls are 5 m high and 17 m apart. The surface of the sand is used for storing pig iron weighing 15 kPa. If the coefficient of the earth pressure at rest is $K_0 = 0.50$, what is the resultant pressure against the walls before and after the application of the surcharge?

Ans. 113 kN/m; 150 kN/m.

5. A smooth vertical wall 6 m high is pushed against a mass of soil having a horizontal surface and an undrained shear strength $s_u = 35$ kPa. The unit weight of the soil is 17 kN/m^3. Its surface carries a uniform load of 10 kPa. What is the total passive Rankine pressure? What is the distance from the base of the wall to the center of pressure? Determine the intensity of lateral pressure at the base of the wall.

Ans. 786 kN/m; 2.61 m; 182 kPa.

6. A smooth vertical wall 4 m high is pushed against an overconsolidated clay ($\gamma = 19$ kN/m^3, $c' = 3$ kPa, $\phi' = 30°$, and average $s_u = 120$ kPa). The water table in the overconsolidated clay is 1 m below the ground surface. The surface of the overconsolidated clay carries a uniform load of 25 kPa. Using the Rankine theory calculate the resultant pressure per lineal meter against the wall when the clay behind the wall fails (a) in an undrained condition, and (b) in a drained condition. (c)