Syntax

Simple Arithmetic Expressions

Here is a BNF grammar for a very simple language of arithmetic expressions:

`t ::=`	*terms*
`true`	*constant true*
`false`	*constant false*
`if t then t else t`	*conditional*
`0`	*constant zero*
`succ t`	*successor*
`pred t`	*predecessor*
`iszero t`	*zero test*

Terminology:

- `t` here is a *metavariable*

Abstract vs. concrete syntax

Q: Does this grammar define a set of *character strings*, a set of *token lists*, or a set of *abstract syntax trees*?

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A: In a sense, all three. But we are primarily interested, here, in abstract syntax trees.

For this reason, grammars like the one on the previous slide are sometimes called *abstract grammars*. An abstract grammar *defines* a set of abstract syntax trees and *suggests* a mapping from character strings to trees.

We then *write* terms as linear character strings rather than trees simply for convenience. If there is any potential confusion about what tree is intended, we use parentheses to disambiguate.

Q: So, are

```
succ 0
succ (0)
(((succ (((((0))))))))
```

"the same term"?

What about

```
succ 0
pred (succ (succ 0))
```

?

A more explicit form of the definition

The set $\mathcal{T}$ of *terms* is the smallest set such that

1. $\{\mathtt{true}, \mathtt{false}, 0\} \subseteq \mathcal{T}$;
2. if $\mathtt{t}_1 \in \mathcal{T}$, then $\{\mathtt{succ\ t}_1, \mathtt{pred\ t}_1, \mathtt{iszero\ t}_1\} \subseteq \mathcal{T}$;
3. if $\mathtt{t}_1 \in \mathcal{T}$, $\mathtt{t}_2 \in \mathcal{T}$, and $\mathtt{t}_3 \in \mathcal{T}$, then
   $\mathtt{if\ t}_1\ \mathtt{then\ t}_2\ \mathtt{else\ t}_3 \in \mathcal{T}$.

Inference rules

An alternate notation for the same definition:

$$\texttt{true} \in \mathcal{T} \qquad \texttt{false} \in \mathcal{T} \qquad \texttt{0} \in \mathcal{T}$$

$$\frac{\texttt{t}_1 \in \mathcal{T}}{\texttt{succ t}_1 \in \mathcal{T}} \qquad \frac{\texttt{t}_1 \in \mathcal{T}}{\texttt{pred t}_1 \in \mathcal{T}} \qquad \frac{\texttt{t}_1 \in \mathcal{T}}{\texttt{iszero t}_1 \in \mathcal{T}}$$

$$\frac{\texttt{t}_1 \in \mathcal{T} \qquad \texttt{t}_2 \in \mathcal{T} \qquad \texttt{t}_3 \in \mathcal{T}}{\texttt{if t}_1 \texttt{ then t}_2 \texttt{ else t}_3 \in \mathcal{T}}$$

Note that "the smallest set closed under..." is implied (but often not stated explicitly).

Terminology:

- axiom vs. rule
- concrete rule vs. rule scheme

Terms, concretely

Define an infinite sequence of sets, $\mathcal{S}_0$, $\mathcal{S}_1$, $\mathcal{S}_2$, ..., as follows:

$$\begin{array}{lcl}
\mathcal{S}_0 & = & \emptyset \\
\mathcal{S}_{i+1} & = & \qquad \{\texttt{true}, \texttt{false}, \texttt{0}\} \\
& & \cup \quad \{\texttt{succ t}_1, \texttt{pred t}_1, \texttt{iszero t}_1 \mid \texttt{t}_1 \in \mathcal{S}_i\} \\
& & \cup \quad \{\texttt{if t}_1 \texttt{ then t}_2 \texttt{ else t}_3 \mid \texttt{t}_1, \texttt{t}_2, \texttt{t}_3 \in \mathcal{S}_i\}
\end{array}$$

Now let

$$\mathcal{S} \quad = \quad \bigcup_i \mathcal{S}_i$$

Comparing the definitions

We have seen two different presentations of terms:

1. as the *smallest* set that is *closed* under certain rules ($\mathcal{T}$)
   - explicit inductive definition
   - BNF shorthand
   - inference rule shorthand
2. as the *limit* ($\mathcal{S}$) of a series of sets (of larger and larger terms)

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What does it mean to assert that "these presentations are equivalent"?

Induction on Syntax

Why two definitions?

The two ways of defining the set of terms are both useful:

1. the definition of terms as the smallest set with a certain closure property is compact and easy to read
2. the definition of the set of terms as the limit of a sequence gives us an *induction principle* for proving things about terms...

Induction on Terms

Definition: The *depth* of a term $\mathtt{t}$ is the smallest i such that $\mathtt{t} \in \mathcal{S}_i$.

From the definition of $\mathcal{S}$, it is clear that, if a term $\mathtt{t}$ is in $\mathcal{S}_i$, then all of its immediate subterms must be in $\mathcal{S}_{i-1}$, i.e., they must have strictly smaller depths.

This observation justifies the *principle of induction on terms*. Let P be a predicate on terms.

> *If, for each term s,*
> *given $P(r)$ for all immediate subterms r of s*
> *we can show $P(s)$,*
> *then $P(t)$ holds for all t.*

Inductive Function Definitions

The set of constants appearing in a term $\mathtt{t}$, written $\mathit{Consts}(\mathtt{t})$, is defined as follows:

$$
\begin{array}{lcl}
\mathit{Consts}(\mathtt{true}) & = & \{\mathtt{true}\} \\
\mathit{Consts}(\mathtt{false}) & = & \{\mathtt{false}\} \\
\mathit{Consts}(\mathtt{0}) & = & \{\mathtt{0}\} \\
\mathit{Consts}(\mathtt{succ}\ \mathtt{t}_1) & = & \mathit{Consts}(\mathtt{t}_1) \\
\mathit{Consts}(\mathtt{pred}\ \mathtt{t}_1) & = & \mathit{Consts}(\mathtt{t}_1) \\
\mathit{Consts}(\mathtt{iszero}\ \mathtt{t}_1) & = & \mathit{Consts}(\mathtt{t}_1) \\
\mathit{Consts}(\mathtt{if}\ \mathtt{t}_1\ \mathtt{then}\ \mathtt{t}_2\ \mathtt{else}\ \mathtt{t}_3) & = & \mathit{Consts}(\mathtt{t}_1) \cup \mathit{Consts}(\mathtt{t}_2) \\
 & & \cup \mathit{Consts}(\mathtt{t}_3)
\end{array}
$$

Simple, right?

First question:

Normally, a “definition” just assigns a convenient name to a previously-known thing. But here, the “thing” on the right-hand side involves the very name that we are “defining”!

So in what sense is this a definition??

Second question: Suppose we had written this instead...

The set of constants appearing in a term $\mathtt{t}$, written $\mathit{BadConsts}(\mathtt{t})$, is defined as follows:

$$
\begin{array}{lcl}
\mathit{BadConsts}(\mathtt{true}) & = & \{\mathtt{true}\} \\
\mathit{BadConsts}(\mathtt{false}) & = & \{\mathtt{false}\} \\
\mathit{BadConsts}(\mathtt{0}) & = & \{\mathtt{0}\} \\
\mathit{BadConsts}(\mathtt{0}) & = & \{\} \\
\mathit{BadConsts}(\mathtt{succ}\ \mathtt{t}_1) & = & \mathit{BadConsts}(\mathtt{t}_1) \\
\mathit{BadConsts}(\mathtt{pred}\ \mathtt{t}_1) & = & \mathit{BadConsts}(\mathtt{t}_1) \\
\mathit{BadConsts}(\mathtt{iszero}\ \mathtt{t}_1) & = & \mathit{BadConsts}(\mathtt{iszero}\ (\mathtt{iszero}\ \mathtt{t}_1))
\end{array}
$$

What is the essential difference between these two definitions?
How do we tell the difference between well-formed inductive definitions and ill-formed ones?
What, exactly, does a well-formed inductive definition mean?

What is a function?

Recall that a *function* f from A (its domain) to B (its co-domain) can be viewed as a two-place *relation* (called the "graph" of the function) with certain properties:

- It is *total*: Every element of its domain occurs at least once in its graph. More precisely:

  For every $a \in A$, there exists some $b \in B$ such that $(a, b) \in f$.

- It is *deterministic*: every element of its domain occurs at most once in its graph. More precisely:

  If $(a, b_1) \in f$ and $(a, b_2) \in f$, then $b_1 = b_2$.

We have seen how to define relations inductively. E.g....
Let *Consts* be the smallest two-place relation closed under the following rules:

$$(\mathtt{true}, \{\mathtt{true}\}) \in \mathit{Consts}$$

$$(\mathtt{false}, \{\mathtt{false}\}) \in \mathit{Consts}$$

$$(\mathtt{0}, \{\mathtt{0}\}) \in \mathit{Consts}$$

$$\frac{(\mathtt{t}_1, C) \in \mathit{Consts}}{(\mathtt{succ}\ \mathtt{t}_1, C) \in \mathit{Consts}}$$

$$\frac{(\mathtt{t}_1, C) \in \mathit{Consts}}{(\mathtt{pred}\ \mathtt{t}_1, C) \in \mathit{Consts}}$$

$$\frac{(\mathtt{t}_1, C) \in \mathit{Consts}}{(\mathtt{iszero}\ \mathtt{t}_1, C) \in \mathit{Consts}}$$

$$\frac{(\mathtt{t}_1, C_1) \in \mathit{Consts} \qquad (\mathtt{t}_2, C_2) \in \mathit{Consts} \qquad (\mathtt{t}_3, C_3) \in \mathit{Consts}}{(\mathtt{if}\ \mathtt{t}_1\ \mathtt{then}\ \mathtt{t}_2\ \mathtt{else}\ \mathtt{t}_3, (\mathit{Consts}(\mathtt{t}_1) \cup \mathit{Consts}(\mathtt{t}_2) \cup \mathit{Consts}(\mathtt{t}_3))) \in \mathit{Consts}}$$

This definition certainly defines a *relation* (i.e., the smallest one with a certain closure property).

Q: How can we be sure that this relation is a *function*?

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A: *Prove it!*

Theorem: The relation *Consts* defined by the inference rules a couple of slides ago is total and deterministic.

I.e., for each term $\mathtt{t}$ there is exactly one set of terms C such that $(\mathtt{t}, C) \in \mathit{Consts}$.

Proof:

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Proof: By induction on $\mathtt{t}$.

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Proof: By induction on $\mathtt{t}$.
To apply the induction principle for terms, we must show, for an arbitrary term $\mathtt{t}$, that if

> for each immediate subterm $\mathtt{s}$ of $\mathtt{t}$, there is exactly one set of terms C_s such that $(\mathtt{s}, C_s) \in \mathit{Consts}$

then

> there is exactly one set of terms C such that $(\mathtt{t}, C) \in \mathit{Consts}$.

Proceed by cases on the form of $\mathtt{t}$.

- If $\mathtt{t}$ is $\mathtt{0}$, $\mathtt{true}$, or $\mathtt{false}$, then we can immediately see from the definition of *Consts* that there is exactly one set of terms C (namely $\{\mathtt{t}\}$) such that $(\mathtt{t}, C) \in \mathit{Consts}$.

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- If $\mathtt{t}$ is $\mathtt{succ}\ \mathtt{t}_1$, then the induction hypothesis tells us that there is exactly one set of terms C_1 such that $(\mathtt{t}_1, C_1) \in \mathit{Consts}$. But then it is clear from the definition of *Consts* that there is exactly one set C (namely C_1) such that $(\mathtt{t}, C) \in \mathit{Consts}$.

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  Similarly when $\mathtt{t}$ is $\mathtt{pred}\ \mathtt{t}_1$ or $\mathtt{iszero}\ \mathtt{t}_1$.

- If $\mathtt{t}$ is $\mathtt{if}\ \mathtt{s}_1\ \mathtt{then}\ \mathtt{s}_2\ \mathtt{else}\ \mathtt{s}_3$, then the induction hypothesis tells us

  - there is exactly one set of terms C_1 such that $(\mathtt{t}_1, C_1) \in \mathit{Consts}$
  - there is exactly one set of terms C_2 such that $(\mathtt{t}_2, C_2) \in \mathit{Consts}$
  - there is exactly one set of terms C_3 such that $(\mathtt{t}_3, C_3) \in \mathit{Consts}$

  But then it is clear from the definition of *Consts* that there is exactly one set C (namely $C_1 \cup C_2 \cup C_3$) such that $(\mathtt{t}, C) \in \mathit{Consts}$.

How about the bad definition?

$$(\texttt{true}, \{\texttt{true}\}) \in \mathit{BadConsts}$$

$$(\texttt{false}, \{\texttt{false}\}) \in \mathit{BadConsts}$$

$$(\texttt{0}, \{\texttt{0}\}) \in \mathit{BadConsts}$$

$$(\texttt{0}, \{\}) \in \mathit{BadConsts}$$

$$\frac{(\texttt{t}_1, C) \in \mathit{BadConsts}}{(\texttt{succ t}_1, C) \in \mathit{BadConsts}}$$

$$\frac{(\texttt{t}_1, C) \in \mathit{BadConsts}}{(\texttt{pred t}_1, C) \in \mathit{BadConsts}}$$

$$\frac{(\texttt{iszero (iszero t}_1\texttt{)}, C) \in \mathit{BadConsts}}{(\texttt{iszero t}_1, C) \in \mathit{BadConsts}}$$

This set of rules defines a perfectly good *relation* — it's just that this relation does not happen to be a function!

Just for fun, let's calculate some cases of this relation...

- For what values of C do we have $(\texttt{false}, C) \in \mathit{BadConsts}$?

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- For what values of C do we have $(\texttt{iszero 0}, C) \in \mathit{BadConsts}$?

Another Inductive Definition

$$
\begin{array}{lcl}
\mathit{size}(\mathtt{true}) & = & 1 \\
\mathit{size}(\mathtt{false}) & = & 1 \\
\mathit{size}(\mathtt{0}) & = & 1 \\
\mathit{size}(\mathtt{succ}\ \mathtt{t}_1) & = & \mathit{size}(\mathtt{t}_1) + 1 \\
\mathit{size}(\mathtt{pred}\ \mathtt{t}_1) & = & \mathit{size}(\mathtt{t}_1) + 1 \\
\mathit{size}(\mathtt{iszero}\ \mathtt{t}_1) & = & \mathit{size}(\mathtt{t}_1) + 1 \\
\mathit{size}(\mathtt{if}\ \mathtt{t}_1\ \mathtt{then}\ \mathtt{t}_2\ \mathtt{else}\ \mathtt{t}_3) & = & \mathit{size}(\mathtt{t}_1) + \mathit{size}(\mathtt{t}_2) + \mathit{size}(\mathtt{t}_3) + 1
\end{array}
$$

Another proof by induction

Theorem: The number of distinct constants in a term is at most the size of the term. I.e., $|\mathit{Consts}(\mathtt{t})| \leq \mathit{size}(\mathtt{t})$.

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There are "three" cases to consider:

Case: $\mathtt{t}$ is a constant

Immediate: $|Consts(\mathtt{t})| = |\{\mathtt{t}\}| = 1 = size(\mathtt{t})$.

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Case: $\mathtt{t}$ is a constant

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Case: $\mathtt{t} = \mathtt{succ}\ \mathtt{t}_1$, $\mathtt{pred}\ \mathtt{t}_1$, or $\mathtt{iszero}\ \mathtt{t}_1$

By the induction hypothesis, $|Consts(\mathtt{t}_1)| \leq size(\mathtt{t}_1)$. We now calculate as follows:
$|Consts(\mathtt{t})| = |Consts(\mathtt{t}_1)| \leq size(\mathtt{t}_1) < size(\mathtt{t})$.

Case: $\mathtt{t} = \mathtt{if}\ \mathtt{t}_1\ \mathtt{then}\ \mathtt{t}_2\ \mathtt{else}\ \mathtt{t}_3$

By the induction hypothesis, $|Consts(\mathtt{t}_1)| \leq size(\mathtt{t}_1)$, $|Consts(\mathtt{t}_2)| \leq size(\mathtt{t}_2)$, and $|Consts(\mathtt{t}_3)| \leq size(\mathtt{t}_3)$. We now calculate as follows:

$$\begin{array}{rcl} |Consts(\mathtt{t})| & = & |Consts(\mathtt{t}_1) \cup Consts(\mathtt{t}_2) \cup Consts(\mathtt{t}_3)| \\ & \leq & |Consts(\mathtt{t}_1)| + |Consts(\mathtt{t}_2)| + |Consts(\mathtt{t}_3)| \\ & \leq & size(\mathtt{t}_1) + size(\mathtt{t}_2) + size(\mathtt{t}_3) \\ & < & size(\mathtt{t}). \end{array}$$