Reasoning about Evaluation

Derivations

We can record the "justification" for a particular pair of terms that are in the evaluation relation in the form of a tree.

(on the board)

Terminology:

- These trees are called *derivation trees* (or just *derivations*).
- The final statement in a derivation is its *conclusion*.
- We say that the derivation is a *witness* for its conclusion (or a *proof* of its conclusion) — it records all the reasoning steps that justify the conclusion.

Observation

Lemma: Suppose we are given a derivation tree $\mathcal{D}$ witnessing the pair $(\mathtt{t}, \mathtt{t}')$ in the evaluation relation. Then either

1. the final rule used in $\mathcal{D}$ is $\textsc{E-IfTrue}$ and we have $\mathtt{t} = \mathtt{if\ true\ then\ t_2\ else\ t_3}$ and $\mathtt{t}' = \mathtt{t_2}$, for some $\mathtt{t_2}$ and $\mathtt{t_3}$, or
2. the final rule used in $\mathcal{D}$ is $\textsc{E-IfFalse}$ and we have $\mathtt{t} = \mathtt{if\ false\ then\ t_2\ else\ t_3}$ and $\mathtt{t}' = \mathtt{t_3}$, for some $\mathtt{t_2}$ and $\mathtt{t_3}$, or
3. the final rule used in $\mathcal{D}$ is $\textsc{E-If}$ and we have $\mathtt{t} = \mathtt{if\ t_1\ then\ t_2\ else\ t_3}$ and $\mathtt{t}' = \mathtt{if\ t_1'\ then\ t_2\ else\ t_3}$, for some $\mathtt{t_1}$, $\mathtt{t_1'}$, $\mathtt{t_2}$, and $\mathtt{t_3}$; moreover, the immediate subderivation of $\mathcal{D}$ witnesses $(\mathtt{t_1}, \mathtt{t_1'}) \in \longrightarrow$.

Induction on Derivations

We can now write proofs about evaluation "by induction on derivation trees."

Given an arbitrary derivation $\mathcal{D}$ with conclusion $\mathtt{t} \longrightarrow \mathtt{t}'$, we assume the desired result for its immediate sub-derivation (if any) and proceed by a case analysis (using the previous lemma) of the final evaluation rule used in constructing the derivation tree.

E.g....

Induction on Derivations — Example

Theorem: If $\mathtt{t} \longrightarrow \mathtt{t}'$, i.e., if $(\mathtt{t}, \mathtt{t}') \in \longrightarrow$, then $\mathit{size}(\mathtt{t}) > \mathit{size}(\mathtt{t}')$.
Proof: By induction on a derivation $\mathcal{D}$ of $\mathtt{t} \longrightarrow \mathtt{t}'$.

1. Suppose the final rule used in $\mathcal{D}$ is E-IfTrue, with $\mathtt{t} = \mathtt{if\ true\ then\ t_2\ else\ t_3}$ and $\mathtt{t}' = \mathtt{t_2}$. Then the result is immediate from the definition of *size*.
2. Suppose the final rule used in $\mathcal{D}$ is E-IfFalse, with $\mathtt{t} = \mathtt{if\ false\ then\ t_2\ else\ t_3}$ and $\mathtt{t}' = \mathtt{t_3}$. Then the result is again immediate from the definition of *size*.
3. Suppose the final rule used in $\mathcal{D}$ is E-If, with $\mathtt{t} = \mathtt{if\ t_1\ then\ t_2\ else\ t_3}$ and $\mathtt{t}' = \mathtt{if\ t_1'\ then\ t_2\ else\ t_3}$, where $(\mathtt{t_1}, \mathtt{t_1'}) \in \longrightarrow$ is witnessed by a derivation $\mathcal{D}_1$. By the induction hypothesis, $\mathit{size}(\mathtt{t_1}) > \mathit{size}(\mathtt{t_1'})$. But then, by the definition of *size*, we have $\mathit{size}(\mathtt{t}) > \mathit{size}(\mathtt{t}')$.

Normal forms

A *normal form* is a term that cannot be evaluated any further — i.e., a term $\mathtt{t}$ is a normal form (or "is in normal form") if there is no $\mathtt{t}'$ such that $\mathtt{t} \longrightarrow \mathtt{t}'$.

A normal form is a state where the abstract machine is halted — i.e., it can be regarded as a "result" of evaluation.

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Recall that we intended the set of *values* (the boolean constants `true` and `false`) to be exactly the possible "results of evaluation." Did we get this definition right?

Values $=$ normal forms

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For the $\Longleftarrow$ direction, it is convenient to prove the contrapositive: If $\mathtt{t}$ is *not* a value, then it is *not* a normal form. The argument goes by induction on $\mathtt{t}$.

Note, first, that $\mathtt{t}$ must have the form $\mathtt{if}\ \mathtt{t}_1\ \mathtt{then}\ \mathtt{t}_2\ \mathtt{else}\ \mathtt{t}_3$ (otherwise it would be a value). If $\mathtt{t}_1$ is $\mathtt{true}$ or $\mathtt{false}$, then rule E-IfTrue or E-IfFalse applies to $\mathtt{t}$, and we are done.

Otherwise, $\mathtt{t}_1$ is not a value and so, by the induction hypothesis, there is some $\mathtt{t}_1'$ such that $\mathtt{t}_1 \longrightarrow \mathtt{t}_1'$. But then rule E-If yields

$$\mathtt{if}\ \mathtt{t}_1\ \mathtt{then}\ \mathtt{t}_2\ \mathtt{else}\ \mathtt{t}_3 \longrightarrow \mathtt{if}\ \mathtt{t}_1'\ \mathtt{then}\ \mathtt{t}_2\ \mathtt{else}\ \mathtt{t}_3$$

i.e., $\mathtt{t}$ is not in normal form.

Numbers

New syntactic forms

`t`	::=	...	*terms*
		`0`	*constant zero*
		`succ t`	*successor*
		`pred t`	*predecessor*
		`iszero t`	*zero test*
`v`	::=	...	*values*
		`nv`	*numeric value*
`nv`	::=		*numeric values*
		`0`	*zero value*
		`succ nv`	*successor value*

New evaluation rules

$$\boxed{\mathtt{t} \longrightarrow \mathtt{t}'}$$

$$\frac{\mathtt{t}_1 \longrightarrow \mathtt{t}_1'}{\mathtt{succ}\ \mathtt{t}_1 \longrightarrow \mathtt{succ}\ \mathtt{t}_1'} \quad \text{(E-Succ)}$$

$$\mathtt{pred}\ \mathtt{0} \longrightarrow \mathtt{0} \quad \text{(E-PredZero)}$$

$$\mathtt{pred}\ (\mathtt{succ}\ \mathtt{nv}_1) \longrightarrow \mathtt{nv}_1 \quad \text{(E-PredSucc)}$$

$$\frac{\mathtt{t}_1 \longrightarrow \mathtt{t}_1'}{\mathtt{pred}\ \mathtt{t}_1 \longrightarrow \mathtt{pred}\ \mathtt{t}_1'} \quad \text{(E-Pred)}$$

$$\mathtt{iszero}\ \mathtt{0} \longrightarrow \mathtt{true} \quad \text{(E-IszeroZero)}$$

$$\mathtt{iszero}\ (\mathtt{succ}\ \mathtt{nv}_1) \longrightarrow \mathtt{false} \quad \text{(E-IszeroSucc)}$$

$$\frac{\mathtt{t}_1 \longrightarrow \mathtt{t}_1'}{\mathtt{iszero}\ \mathtt{t}_1 \longrightarrow \mathtt{iszero}\ \mathtt{t}_1'} \quad \text{(E-IsZero)}$$

Values are normal forms

Our observation a few slides ago that all values are in normal form still holds for the extended language.

Is the converse true? I.e., is every normal form a value?

Values are normal forms, but we have stuck terms

Our observation a few slides ago that all values are in normal form still holds for the extended language.

Is the converse true? I.e., is every normal form a value?
No: some terms are *stuck*.

Formally, a stuck term is one that is a normal form but not a value.
What are some examples?

Stuck terms model run-time errors.

Multi-step evaluation.

The *multi-step evaluation* relation, $\longrightarrow^*$, is the reflexive, transitive closure of single-step evaluation.

I.e., it is the smallest relation closed under the following rules:

$$\frac{\mathtt{t} \longrightarrow \mathtt{t}'}{\mathtt{t} \longrightarrow^* \mathtt{t}'}$$

$$\mathtt{t} \longrightarrow^* \mathtt{t}$$

$$\frac{\mathtt{t} \longrightarrow^* \mathtt{t}' \qquad \mathtt{t}' \longrightarrow^* \mathtt{t}''}{\mathtt{t} \longrightarrow^* \mathtt{t}''}$$

Termination of evaluation

Theorem: For every $\mathtt{t}$ there is some normal form $\mathtt{t}'$ such that $\mathtt{t} \longrightarrow^* \mathtt{t}'$.

Proof:

Termination of evaluation

Theorem: For every $\mathtt{t}$ there is some normal form $\mathtt{t}'$ such that $\mathtt{t} \longrightarrow^* \mathtt{t}'$.

Proof:

- First, recall that single-step evaluation strictly reduces the size of the term:

$$\textit{if } t \longrightarrow t'\textit{, then } \textit{size}(t) > \textit{size}(t')$$

- Now, assume (for a contradiction) that

$$t_0,\ t_1,\ t_2,\ t_3,\ t_4,\ \ldots$$

  is an infinite-length sequence such that

$$t_0 \longrightarrow t_1 \longrightarrow t_2 \longrightarrow t_3 \longrightarrow t_4 \longrightarrow \cdots.$$

- Then

$$\textit{size}(t_0) > \textit{size}(t_1) > \textit{size}(t_2) > \textit{size}(t_3) > \ldots$$

- But such a sequence cannot exist — contradiction!

Termination Proofs

Most termination proofs have the same basic form:

Theorem: *The relation* $R \subseteq X \times X$ *is terminating — i.e., there are no infinite sequences* x_0, x_1, x_2, *etc. such that* $(x_i, x_{i+1}) \in R$ *for each* i.

Proof:

1. *Choose*
   - *a well-founded set* $(W, <)$ *— i.e., a set* W *with a partial order* $<$ *such that there are no infinite descending chains* $w_0 > w_1 > w_2 > \ldots$ *in* W
   - *a function* f *from* X *to* W
2. *Show* $f(x) > f(y)$ *for all* $(x, y) \in R$
3. *Conclude that there are no infinite sequences* x_0, x_1, x_2, *etc. such that* $(x_i, x_{i+1}) \in R$ *for each* i, *since, if there were, we could construct an infinite descending chain in* W.