EE469: Feedback Control Systems for Mechanical Engineers
Lecture notes set 20
Nyquist Stability Criterion – Continued. Gain Margin and Phase Margin. Stability via Bode Plots

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Range of gain for stability via the Nyquist Criterion

- Since the gain is simply a multiplying factor, the effect of the gain is to multiply the resultant by a constant anywhere along the Nyquist diagram. Thus, as the gain is varied, the Nyquist diagram expands (increased gain) or shrinks (decreased gain). These motions could move the Nyquist diagram past the -1 point, changing the stability picture.

- From another perspective, we can think of the Nyquist diagram as remaining stationary, and the -1 point moving along the real axis. In order to do this, we set the position of the critical point at $-1/K$ rather than -1.

- If the open-loop system contains a variable gain, K, one should set $K = 1$, draw the Nyquist diagram, consider the critical point to be at $-1/K$, and then adjust the value of K to yield stability.

Example

Consider the unity feedback system with the open-loop transfer function

$$G(s) = \frac{K}{s(s+3)(s+5)}.$$

Problem. Using Nyquist criterion, find the range of gain for stability, instability, and the value of gain for marginal stability.

Solution.

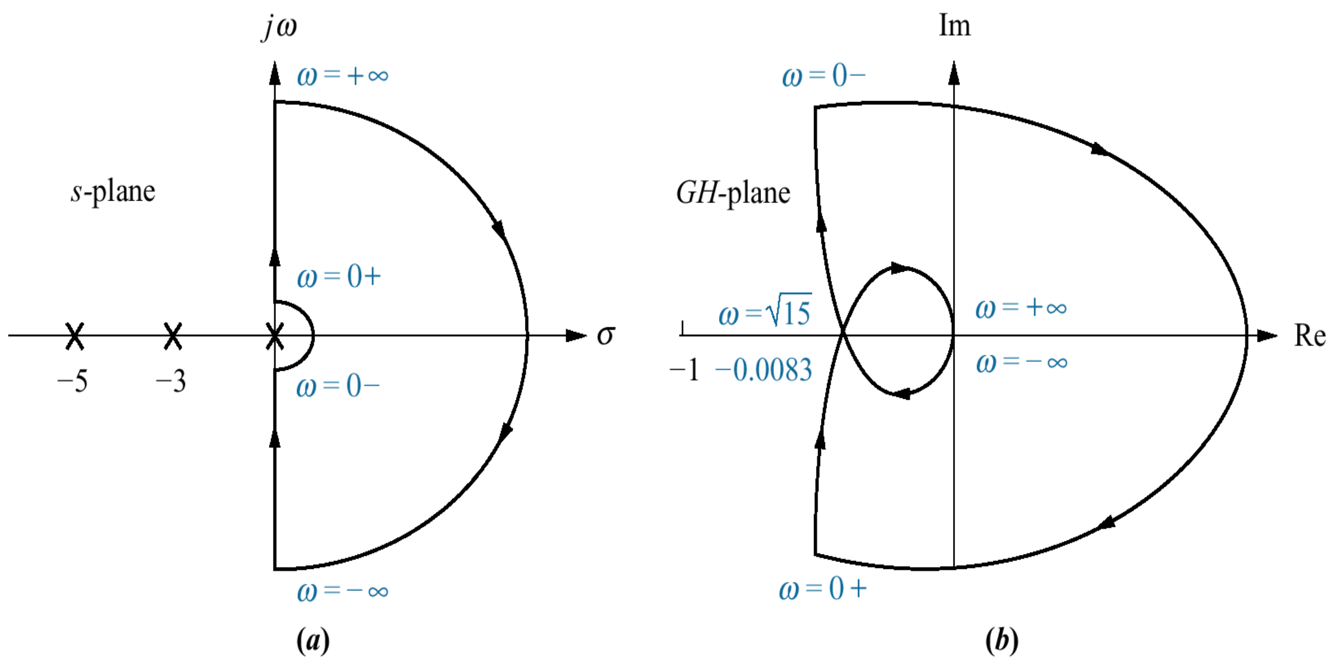
- For the points of imaginary axis,

$$G(\mathrm{j}\omega)H(\mathrm{j}\omega) = \frac{1}{\mathrm{j}\omega(\mathrm{j}\omega+3)(\mathrm{j}\omega+5)} = \frac{-8\omega^2 - j\left(15\omega - \omega^3\right)}{64\omega^4 + \omega^2(15-\omega^2)^2}. \tag{1}$$

- Find the point where the Nyquist diagram intersects the negative part of real axis. Setting the imaginary part in (1) equal to zero, we get $\omega = \sqrt{15}$. Substituting this value of ω into (1), we get the real part $= -0.0083$.

- The open-loop system is stable ($P = 0$), therefore, for stability of the closed-loop system ($Z = 0$), N must be equal to zero, i.e. the critical point must be outside the contour. Thus, K can be increased by $1/0.0083 = 120.48$, before the Nyquist diagram encircles -1. Thus, if $K < 120.48$, the system is stable. If $K = 120.48$, the system is marginally stable, and for $K > 120.48$, the closed-loop system is unstable.

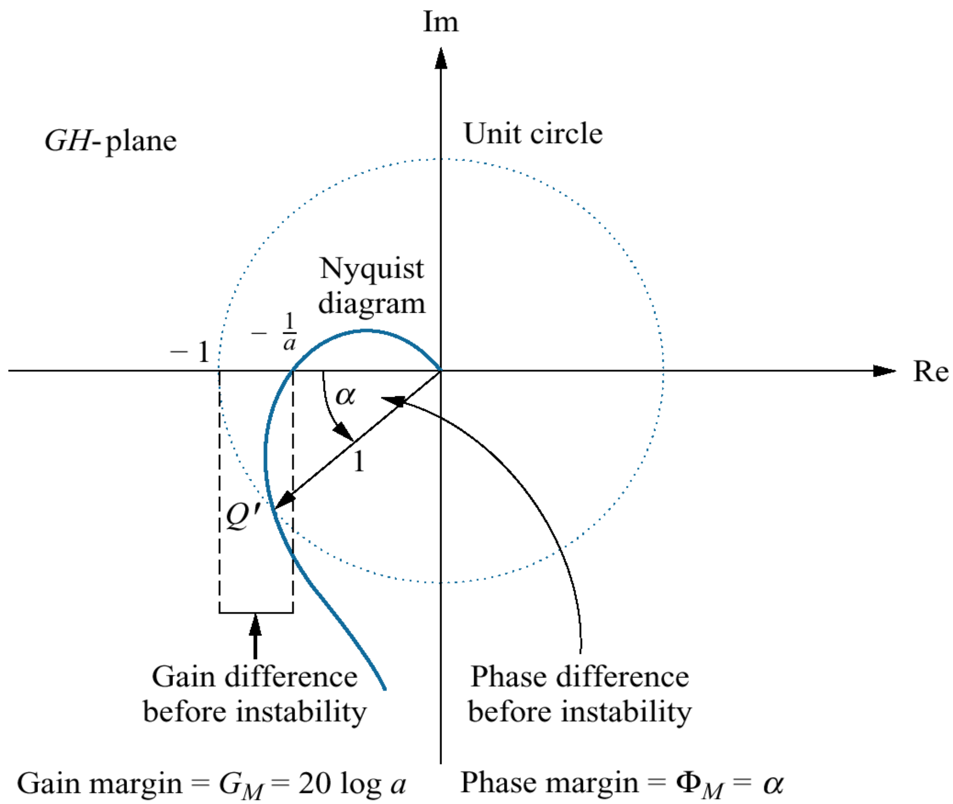
Nyquist contour and Nyquist diagram for Example 1



Gain Margin and Phase Margin

Gain Margin, G_m, is the change in open-loop gain expressed in decibels (dB), required at 180° of phase shift to make the closed-loop system unstable.

Phase margin, Φ_M, is the change in open-loop phase shift required at unity gain to make the closed-loop system unstable.



Example 2. Finding gain and phase margin. Consider the unity negative feedback system with the open loop transfer function

$$G(s) = \frac{K}{(s^2+2s+2)(s+2)}.$$

Problem. Find gain and phase margin, if $K = 6$.

Solution. First, find the frequency, where the Nyquist diagram crosses the negative real axis. We have

$$G(\mathrm{j}\omega)H(\mathrm{j}\omega) = \frac{6}{((\mathrm{j}\omega)^2 + 2\,\mathrm{j}\omega + 2)(\mathrm{j}\omega + 2)} = \frac{6[4(1-\omega^2) - \mathrm{j}\omega(6-\omega^2)]}{16(1-\omega^2)^2 + \omega^2(6-\omega^2)^2}.$$

Thus, the Nyquist diagram crosses the negative real axis at $\omega_0 = \sqrt{6}$. At this frequency, $G(\jmath\omega_0) = -0.3$. Thus, the gain can be increased by $(1/0.3) = 3.33$ before the real part becomes -1. Thus, the gain margin is

$$G_m = 20\log 3.33 \approx 10.45dB.$$

To find phase margin, one should calculate the frequency, where $G(\mathrm{j}\omega) = -1$. It requires some numerical calculation. Below, we will see, that the problem can be solved easily by using Bode plots.●

Stability via Bode Plots. Example

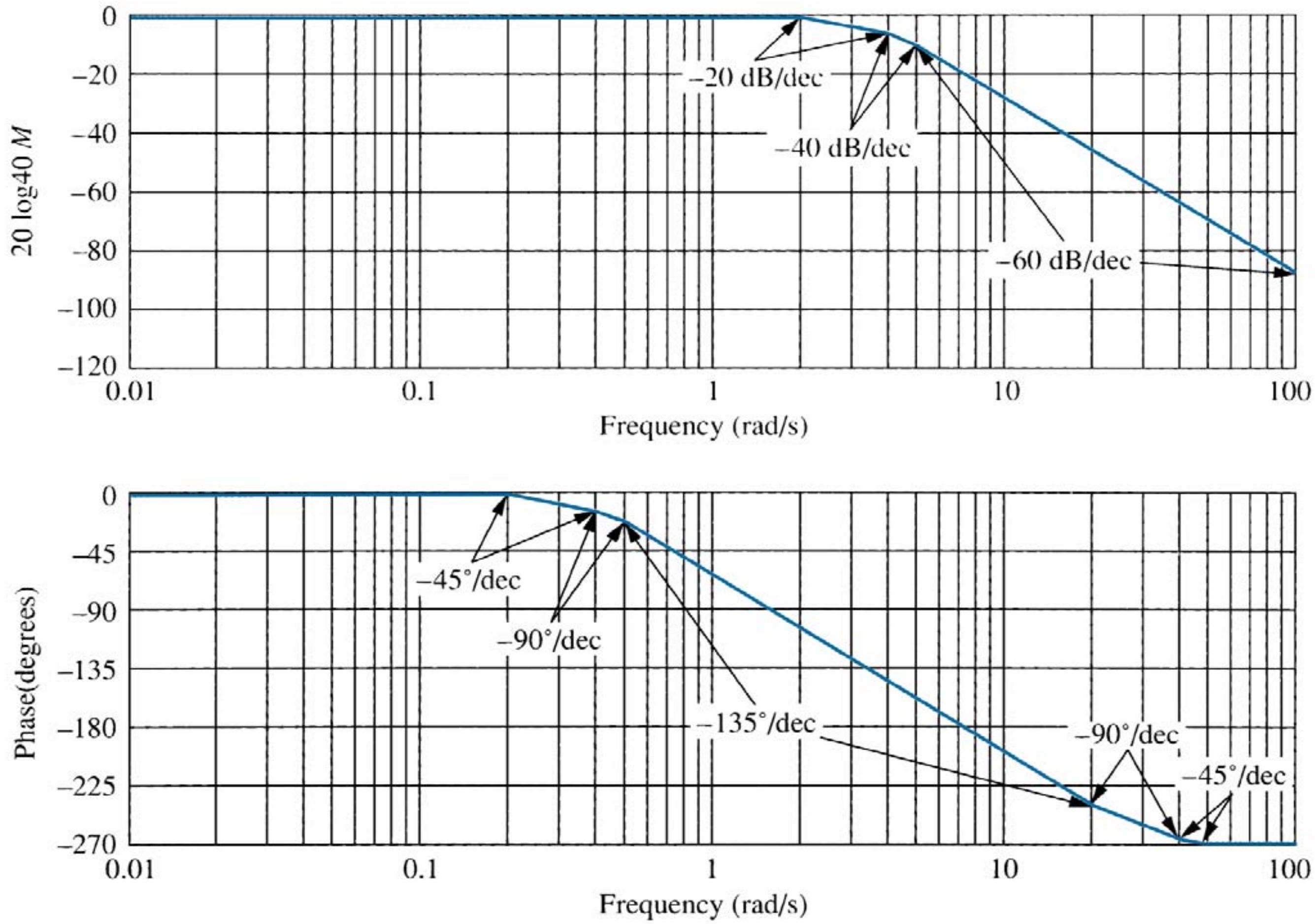
Consider the unity feedback system with

$$G(s) = \frac{K}{(s+2)(s+4)(s+5)}.$$

Problem

- Find the range of K within which the closed-loop system is stable.
- For $K = 200$, find the gain margin and the phase margin.

Bode plots for system of Example 3



Solution.

- Since the low frequency gain of the open loop system is $K/40$, we let $K = 40$ so that the log-magnitude plot starts at $0dB$.

- The open loop system is stable. Therefore, the closed-loop system is stable if the frequency response has a gain less than unity when the phase is $180°$.

- We see that at a frequency $\omega = 7$, when the phase is $180°$, the magnitude plot is $-20dB$. Therefore the system is stable for $K = 40$.

- Increase in gain of $+20dB$ is possible before the system becomes unstable. Therefore, we have
$$20 \log \left(\frac{K_{\max}}{40} \right) = 20(dB),$$
where $K_{\max}$ is maximal gain so that the closed-loop system is stable. Thus, $\log \left(K_{\max}/40 \right) = 1$, i.e. $K_{\max} = 40 \times 10 = 400$. The system is stable for $0 < K < 400$.

- To find the gain and phase margin for $K = 200$, note that the magnitude plot will be $20\log(200/40) = 20\log 5 = 13.98dB$ higher. As we already find, at a frequency $\omega = 7$, when the phase is $180°$, the magnitude plot is $-20dB$. Therefore, the gain margin is $-20 + 13.98 = 6.02(dB)$.

- To find the phase margin, we look on the magnitude plot for the frequency, where the gain is $0dB$. Since the magnitude plot is $13.98dB$ lower than the actual plot, the $0dB$ crossing (i.e. $-13.98dB$ crossing for the plot in figure) occurs at $\omega = 5.5$. At this frequency, the phase angle is $-165\circ$. Therefore, the phase margin is $-165° - (-180°) = 15°$.