AUX EXTRA

P3 Nos dan la densidad de Laplace.

$$f_X(x) = \frac{1}{2b} e^{-\frac{|x-\mu|}{b}} \qquad \forall x \in \mathbb{R}$$

(a) Calculemos la f.g.m. asociada

$$M_X(t) = \mathbb{E}(e^{tX}) = \int_{-\infty}^{+\infty} e^{tx} \frac{1}{2b} e^{-\frac{|x-\mu|}{b}} \, dx$$

$$= \underbrace{\int_{\mu}^{+\infty} \frac{e^{tx} e^{-\frac{(x-\mu)}{b}}}{2b} \, dx}_{x-\mu>0} + \underbrace{\int_{-\infty}^{\mu} \frac{e^{tx} e^{\frac{(x-\mu)}{b}}}{2b} \, dx}_{x-\mu<0}$$

$$= \frac{1}{2b}\left(e^{\mu/b} \int_{\mu}^{+\infty} e^{(t-\frac{1}{b})x} \, dx + e^{-\mu/b} \int_{-\infty}^{\mu} e^{(t+\frac{1}{b})x} \, dx \right)$$

Si $t-\frac{1}{b}<0$ 　　　　 Si $(t+\frac{1}{b})>0$

$$= \frac{1}{2b}\left(\frac{e^{\mu/b}}{t-\frac{1}{b}} \cdot -e^{(t-\frac{1}{b})\mu} + \frac{e^{-\mu/b}}{t+\frac{1}{b}} \cdot e^{(t+\frac{1}{b})\mu} \right)$$

$$= \frac{e^{\mu t}}{2b}\left(-\frac{1}{t-\frac{1}{b}} + \frac{1}{t+\frac{1}{b}} \right)$$

$$= \frac{e^{\mu t}}{2}\left(-\frac{1}{bt-1} + \frac{1}{bt+1} \right) = \frac{e^{\mu t}}{1-b^2t^2} \qquad \text{Si } |t| < \frac{1}{b}$$

(b) $\mathbb{E}(X) = M_X^{(1)}(0) = \left(\frac{e^{\mu t}}{1-b^2t^2}\right)'\Big|_{t=0}$

$$= \frac{\mu e^{\mu t}(1-b^2t^2) + e^{\mu t}\cdot 2t\,b^2}{(1-b^2t^2)^2}\Big|_{t=0}$$

$$= \mu$$

$Var(X) = \mathbb{E}(X^2) - \mathbb{E}^2(X) = \mathbb{E}(X^2) - \mu^2$

$$\mathbb{E}(X^2) = M_X^{(2)}(0) = \left(\frac{(\mu - \mu\cancel{b}^2t^2 + 2t\,b^2)e^{\mu t}}{(1-b^2t^2)^2}\right)'\Big|_{t=0}$$

$$= \Big\{\Big[\big(\mu e^{\mu t}(\mu - \mu b^2t^2 + 2tb^2) + e^{\mu t}(-2t\mu b^2 + 2b^2)\Big](1-b^2t^2)^2$$

$$- (\mu - \mu b^2t^2 + 2t\,b^2)e^{\mu t}\cdot 2(1-b^2t^2)(-2t\,b^2)\Big\}$$

$$\cdot \frac{1}{(1-b^2t^2)^4}\Big|_{t=0}$$

$$= \left[\frac{\mu^2 + 2b^2}{1} - 0\right] = \mu^2 + 2b^2$$

$\Rightarrow Var(X) = \mathbb{E}(X^2) - \mathbb{E}^2(X) = \mu^2 + 2b^2 - \mu^2 = 2b^2$

(c) $\mu = 0$ Nos piden $f_{|X|}(x)$

$\Downarrow$

$$f_X(x) = \frac{1}{2b} e^{-\frac{|x|}{b}} \qquad \forall x \geq 0$$

$$F_{|X|}(x) = \mathbb{P}(|X| \leq x) = \mathbb{P}(-x \leq X \leq x)$$

función par

$$= \int_{-x}^{x} f_X(u)\,du = \int_{-x}^{x} \frac{1}{2b} e^{-\frac{|u|}{b}} du = \frac{2}{2b} \int_0^x e^{-\frac{u}{b}} du$$

$$= \frac{2}{2b} \cdot -b \; e^{-\frac{u}{b}} \Big|_0^x = -\left(e^{-\frac{x}{b}} - 1\right)$$

$$= 1 - e^{-\frac{x}{b}}$$

Si $Y = |X|$ $f_{|X|}(x) = 0 \quad \forall x < 0.$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left(1 - e^{-y/b}\right) = \frac{1}{b} e^{-y/b}.$$

$$\Rightarrow Y \sim \exp\left(\frac{1}{b}\right)$$

d) Veamos cual es la f.g.m de $Z = \lambda_1 Y_1 - \lambda_2 Y_2$

$$M_Z(t) = \mathbb{E}(e^{tZ}) = \mathbb{E}\left(e^{t\lambda_1 Y_1} e^{-t\lambda_2 Y_2}\right)$$

$$\underset{\text{Indep.}}{=} \mathbb{E}(e^{t\lambda_1 Y_1})\, \mathbb{E}\left(e^{t\cdot(-\lambda_2 Y_2)}\right)$$

$$\mathbb{E}(e^{t\lambda_1 Y_1}) = \int_{-\infty}^{+\infty} e^{t\lambda_1 y} f_{Y_1}(y)\, dy$$

$$= \int_{-\infty}^{+\infty} e^{t\lambda_1 y} \lambda_1 e^{-\lambda_1 y} \mathbb{1}_{[0,+\infty)}(y)\, dy$$

$$= \lambda_1 \int_0^{+\infty} e^{(\lambda_1 t - \lambda_1) y}\, dy$$

$$= \frac{\lambda_1}{\lambda_1 t - \lambda_1} e^{(\lambda_1 t - \lambda_1) y}\Big|_0^{+\infty}$$

$$\text{si } t < 1$$

$$= \frac{-1}{t-1} = \frac{1}{1-t}$$

Análogamente

$$\mathbb{E}(e^{-t\lambda_2 Y_2}) = \lambda_2 \int_0^{+\infty} e^{-(\lambda_2 t + \lambda_2) y}\, dy$$

$$= \frac{1}{-(t+1)} e^{-(\lambda_2 t + \lambda_2) y}\Big|_0^{+\infty} \overset{t > -1}{=} \frac{1}{t+1}$$

$$\therefore M_Z(t) \overset{|t|<1}{=} \mathbb{E}(e^{t\lambda_1 Y_1})\,\mathbb{E}(e^{-\lambda_2 t Y_2})$$

$$= \frac{1}{1-t^2}$$

Como la función g. m. caracteriza una distribución

$$\Rightarrow Z \sim \text{Laplace}(\underset{\mu}{0}, \underset{b}{1})$$

$\boxed{4}$ $g : [1, +\infty)^2 \longrightarrow \{(u,v) : 0 < \frac{1}{u} \leq v \leq u\}$.

(a) $g(x,y) = \begin{pmatrix} xy \\ \frac{x}{y} \end{pmatrix} = \begin{pmatrix} U \\ V \end{pmatrix}$

$$J_g(x,y) = \det \begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{pmatrix} = \det \begin{pmatrix} y & x \\ \frac{1}{y} & -\frac{x}{y^2} \end{pmatrix} = -\frac{x}{y} \cdot 2$$

$$\downarrow$$

$$= -2v$$

$$\begin{matrix} U = xy \\ V = \frac{X}{Y} \end{matrix} \implies \begin{matrix} X = \sqrt{UV} \\ Y = \sqrt{\frac{U}{V}} \end{matrix}$$

$$g^{-1} : (u,v) \longmapsto \begin{pmatrix} \sqrt{uv} \\ \sqrt{\frac{u}{v}} \end{pmatrix}$$

$$\Rightarrow f_{U,V}(u,v) = f_{X,Y}(g^{-1}(u,v)) \cdot |J_g(g^{-1}(u,v))|^{-1}$$

$$= \frac{1}{uv \cdot \frac{u}{v}} \cdot |-2v|^{-1} = \frac{1}{2u^2 v}$$

Esto siempre y cuando que $\sqrt{uv} \geq 1$, $\sqrt{\frac{u}{v}} \geq 1$, $u, v > 0$.

$$\iff uv \geq 1 \quad y \quad \frac{u}{v} \geq 1$$

$$\iff \underbrace{u \geq \frac{1}{v}}_{v \geq \frac{1}{u}} \quad y \quad u \geq v. \iff 0 \leq \frac{1}{u} \leq v \leq u$$

En cualquier otro caso $f_{U,V}(u,v) = 0$

b)

$$f_U(u) = \int_{-\infty}^{+\infty} f_{U,V}(u,v)\, dv$$

[redacted] $\forall\, u \geq 1$

~~[illegible]~~

Así $[1/u, u] \neq \phi$.

$$= \int_{-\infty}^{+\infty} \frac{1}{2u^2 v} \cdot \mathbb{1}^{(u,v)}_{\{(u,v):\ 0 \leq \frac{1}{u} \leq v \leq u\}}\, dv$$

$$= \int_{1/u}^{u} \frac{1}{2u^2 v}\, dv =$$

$$= \frac{1}{2u^2} \ln(v) \Big|_{1/u}^{u} = \frac{1}{2u^2}\left(\ln(u) + \ln(u)\right)$$

$$= \frac{2\ln(u)}{2u^2} = \frac{\ln(u)}{u^2} \quad \forall u \geq$$

$$\Rightarrow f_U(u) = \begin{cases} \frac{\ln u}{u^2} & u \geq 1 \\ 0 & \sim \end{cases}$$

$$f_V(v) = \int_{-\infty}^{+\infty} f_{U,V}(u,v)\, du = \int_{-\infty}^{+\infty} \frac{1}{2u^2 v}\, \mathbb{1}^{(u,v)}_{\{(u,v):\ 0 \leq \frac{1}{u} \leq v \leq u\}}\, du$$

Esto siempre que $u \geq v$ o $\frac{1}{u} \leq v$ $\left(\Leftrightarrow u \geq \frac{1}{v}\right)$

$\Rightarrow u \geq \max\{v, \frac{1}{v}\}$ $\forall v$

$$= \int_{\max\{v, \frac{1}{v}\}}^{+\infty} \frac{1}{2u^2 v}\, du = \frac{1}{2v} \int_{\max\{v, \frac{1}{v}\}}^{+\infty} \frac{1}{u^2}\, du = -\frac{1}{2v} \cdot \frac{1}{u} \Big|_{\max\{v, \frac{1}{v}\}}^{+\infty}$$

$$= \cancel{\frac{\max\{v, \frac{1}{v}\}}{2v}} \quad \frac{1}{2v \cdot \max\{v, \frac{1}{v}\}}$$

$$\Rightarrow f_V(v) = \begin{cases} \frac{1}{2v \max\{v, \frac{1}{v}\}} & v > 0 \\ 0 & \sim \end{cases}$$

$$\Rightarrow \quad f_V(v) = \begin{cases} \frac{1}{2} & 0 < v < 1 \\ \frac{1}{2v^2} & v \geq 1 \\ 0 & v \leq 0 \end{cases}$$

P5

$$f_{X,Y}(x,y) = \begin{cases} \frac{e^{-x/y} e^{-y}}{y} & 0 < x < +\infty \;,\; 0 < y < +\infty \\ 0 & \sim \end{cases}$$

Se sabe que

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\frac{e^{-x/y} e^{-y}}{y}}{\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx}$$

$$= \frac{e^{-x/y} e^{-y}}{\cancel{y}} \cdot \frac{\cancel{y}}{\cancel{e^{-y}} \int_0^{+\infty} e^{-x/y}\,dx} = \frac{e^{-x/y}}{-y\, e^{-\frac{x}{y}} \Big|_0^{+\infty}}$$

$$= \frac{e^{-\frac{x}{y}}}{y}.$$

$$\Rightarrow \mathbb{P}(X > 1 \mid Y = y) = \int_1^{+\infty} f_{X|Y=y}(x|y)\,dx$$

$$= \int_1^{+\infty} \frac{e^{-\frac{x}{y}}}{y}\,dx = \frac{1}{y} \cdot -y\, e^{-\frac{x}{y}} \Big|_1^{+\infty}$$

$$= \boxed{e^{-\frac{1}{y}}}$$