Input Signals (Ch. 5)

The quality of the estimated model depends on the choice of input signal.

Examples:

- Step function
- Pseudo-random binary sequences (PRBS)
- Autoregressive moving average process (ARMA)
- Sum of sinusoids.

Most often the input signal is characterized by its first and second moments:

$$\begin{cases} m = E[u(t)] \\ r(\tau) = E\left[(u(t) - m)(u(t) - m)^T\right] \end{cases}$$

and/or its spectral density:

$$\phi(\omega) = \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} r(\tau) e^{-i\tau\omega}$$

Rem. for stationary signals

$$m = \frac{1}{N} \sum_{t=1}^{N} u(t)$$

Pseudo-Random Binary Sequences (PRBS)

A PRBS $(u(t))_t$ is a periodic, deterministic signal with white noise-like properties.

$$u(t) = \operatorname{rem}\left(A(q^{-1})e(t), 2\right)$$

Properties

- The signal takes values $\{0, 1\}$ in a fashion dictated by A.
- Spectral properties are determined by $A(q)$ and in particular by the period length $M = 2^n - 1$.
- Deterministic sequence behaving as noise (reproducible).

ARMA Process

$$A(q^{-1})y(t) = C(q^{-1})e(t)$$

where $e(t)$ is white noise with $E[e(t)] = 0$ and $E[e(t)e(s)] = \lambda^2\delta_{ts}$.

Properties

- The signal $u(t)$ can be obtained by filtering $e(t)$.
- The filters (A, C) can be tuned to possess (almost) any frequency characteristics.
- The spectral density of an ARMA process $y(t)$ is given as

$$\phi_y(\omega) = \frac{\lambda^2}{2\pi}\left|\frac{C(e^{i\omega})}{A(e^{i\omega}}\right|^2$$

Persistent Excitation

In order to obtain a good estimate of a (parametric) model, the input signal has to be 'rich' enough so that all 'modes' of the system are excited.

An inout is said to be persistently exciting (PE) if:

- The following limit exists for all τ

$$r_u(\tau) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N-\tau} u(t+\tau)u^T(t)$$

Rem. $u(t)$ ergodic implies that for any t

$$r_u(\tau) = E[u(t+\tau)u^T(t)]$$

- The matrix $\mathbf{R}_u(n)$

$$\mathbf{R}_u = \begin{bmatrix} r_u(0) & r_u(1) & \dots & r_u(n-1) \\ r_u(1) & r_u(0) & \dots & \vdots \\ \vdots & & \ddots & \\ r_u(n-1) & \dots & & r_u(0) \end{bmatrix}$$

is positive (strictly) definite.

- Or, $\det(\mathbf{R}_u(n)) \neq 0$.

- Or $u(t)$ is PE of order n if $\phi_u(\omega) \neq 0$ on at least n points on the interval $-\pi < \omega < \pi$.

An input signal is PE of order $2n$ can be used to consistently estimate parameters of a model of order $\leq n$.

- A step function that is PE of order 1
- A PRBS with period M is PE of order M.
- An ARMA process is PE of any finite order.
- A sum of m sinusoids is PE of order $2M$ (if $\omega_m \neq 0, -\pi, \pi$)

Another important observation!

A parametric model becomes more accurate in the frequency region where the input signal has a major part of its energy.

A physical process is often of low frequency character $\rightarrow$ use low-pass filtered signal as input.