According to Equation (12-29), the inverse Laplace transform is given by the integral formula

$$f(t) = \mathcal{L}^{-1}(F(s)) = \frac{1}{2\pi i}\int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} F(s)\, e^{st} ds,$$

where σ_0 is any suitably chosen large positive constant. This improper integral is a contour integral taken along the vertical line $s = \sigma_0 + i\tau$ in the complex $s = \sigma + i\tau$ plane. We use the residue theory in Chapter 8 to evaluate it. We leave the cases in which the integrand has either infinitely many poles or branch points for you to research in advanced texts. We state the following more elementary theorem.

▶ Theorem 12.22 (Inverse Laplace transform) *Let* $F(s) = \frac{P(s)}{Q(s)}$*, where* $P(s)$ *and* $Q(s)$ *are polynomials of degree* m *and* n*, respectively, and* $n > m$*. The inverse Laplace transform* $F(s)$ *is* $f(t)$*, which is given by*

$$f(t) = \mathcal{L}^{-1}(F(s)) = \Sigma \mathrm{Res}\left[F(s)\, e^{st}, s_k\right], \tag{12-43}$$

where the sum is taken over all of the residues of the complex function $F(s)\, e^{st}$.

Proof Let σ_0 be chosen so that all the poles of $F(s)\, e^{st}$ lie to the left of the vertical line $s = \sigma_0 + i\tau$. Let Γ_R denote the contour consisting of the vertical line segment between the points $\sigma_0 \pm iR$ and the left semicircle $C_R : s = \sigma_0 + \mathrm{Re}^{i\theta}$, where $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$, as shown in Figure 12.27. A slight modification of the proof of Jordan's lemma reveals that

$$\lim_{R\to+\infty} \int_{C_R} \frac{P(s)}{Q(s)} e^{st} ds = 0.$$

We now use the residue theorem to get

$$\mathcal{L}^{-1}(F(s)) = \lim_{R\to+\infty} \frac{1}{2\pi i}\int_{\Gamma_R} \frac{P(s)}{Q(s)} e^{st} ds = \Sigma \mathrm{Res}\left[F(s)\, e^{st}, s_k\right],$$

and the proof of the theorem is complete.

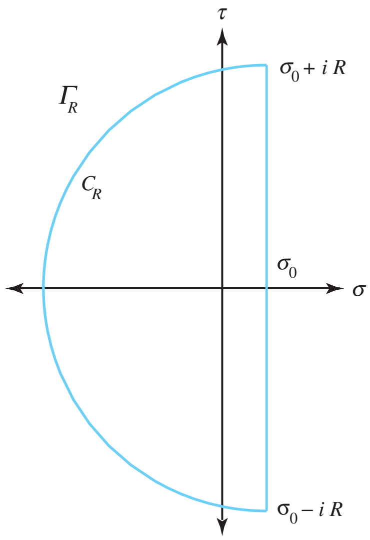

Figure 12.27 The contour Γ_R.