Ejercicio: Sean $a, b, c \in \mathbb{R}$ con $c \neq 0$. Aplicando transformada de Laplace demuestre la igualdad:

$$e^{at} * e^{bt}\text{sen}(ct) = \frac{c}{c^2+(b-a)^2}[e^{at} - e^{bt}\cos(ct)] + \frac{b-a}{c^2+(b-a)^2}e^{bt}\text{sen}(ct).$$

Solución:

Aplicando transformada de Laplace a cada termino del lado derecho de la expresión se tiene:

$$\begin{aligned}
\mathcal{L}\left(\frac{c}{c^2+(b-a)^2}[e^{at} - e^{bt}\cos(ct)]\right)(s) &= \frac{c}{c^2+(b-a)^2}\mathcal{L}(e^{at} - e^{bt}\cos(ct))(s) \\
&= \frac{c}{c^2+(b-a)^2}(\mathcal{L}(e^{at})(s) - \mathcal{L}(e^{bt}\cos(ct))(s)) \\
&= \frac{c}{c^2+(b-a)^2}\left(\frac{1}{s-a} - \mathcal{L}(\cos(ct))(s-b)\right) \\
&= \frac{c}{c^2+(b-a)^2}\left(\frac{1}{s-a} - \frac{s-b}{(s-b)^2+c^2}\right) \\
&= \frac{c(b^2+c^2-bs+as-ab)}{(c^2+(b-a)^2)(s-a)((s-b)^2+c^2)}
\end{aligned}$$

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$$\begin{aligned}
\mathcal{L}\left(\frac{b-a}{c^2+(b-a)^2}e^{bt}\text{sen}(ct)\right)(s) &= \frac{b-a}{c^2+(b-a)^2}\mathcal{L}(e^{bt}\text{sen}(ct))(s) \\
&= \frac{b-a}{c^2+(b-a)^2}\mathcal{L}(\text{sen}(ct))(s-b) \\
&= \frac{b-a}{c^2+(b-a)^2}\frac{c}{(s-b)^2+c^2}.
\end{aligned}$$

Sumando ambas expresiones nos resulta:

$$\begin{aligned}
\mathcal{L}\left(\frac{c}{c^2+(b-a)^2}[e^{at} - e^{bt}\cos(ct)] + \frac{b-a}{c^2+(b-a)^2}e^{bt}\text{sen}(ct)\right)(s) &= \frac{c(c^2+(a-b)^2)}{(c^2+(b-a)^2)(s-a)((s-b)^2+c^2)} \\
&= \frac{c}{(s-a)((s-b)^2+c^2)} \\
&= \frac{1}{s-a}\frac{c}{(s-b)^2+c^2} \\
&= \mathcal{L}(e^{at})(s)\mathcal{L}(e^{bt}\text{sen}(ct))(s) \\
&= \mathcal{L}(e^{at} * e^{bt}\text{sen}(ct))(s)
\end{aligned}$$

Aplicando el Teorema de Lerch, se deduce la igualdad:

$$e^{at} * e^{bt}\text{sen}(ct) = \frac{c}{c^2+(b-a)^2}[e^{at} - e^{bt}\cos(ct)] + \frac{b-a}{c^2+(b-a)^2}e^{bt}\text{sen}(ct).$$