4.10. For extensions of duality theory to problems involving general convex functions and constraint sets, see Rockafellar (1970) and Bertsekas (1995b).

4.12 Exercises 4.6 and 4.7 are adapted from Boyd and Vandenberghe (1995). The result on strict complementary slackness (Exercise 4.20) was proved by Tucker (1956). The result in Exercise 4.21 is due to Clark (1961). The result in Exercise 4.30 is due to Helly (1923). Input-output macroeconomic models of the form considered in Exercise 4.32, have been introduced by Leontief, who was awarded the 1973 Nobel prize in economics. The result in Exercise 4.41 is due to Carathéodory (1907).

Chapter 5

Sensitivity analysis

Contents

Consider the standard form problem

$$\begin{array}{ll} \text{minimize} & \mathbf{c}'\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}, \end{array}$$

and its dual

$$\begin{array}{ll} \text{maximize} & \mathbf{p}'\mathbf{b} \\ \text{subject to} & \mathbf{p}'\mathbf{A} \leq \mathbf{c}'. \end{array}$$

In this chapter, we study the dependence of the optimal cost and the optimal solution on the coefficient matrix $\mathbf{A}$, the requirement vector $\mathbf{b}$, and the cost vector $\mathbf{c}$. This is an important issue in practice because we often have incomplete knowledge of the problem data and we may wish to predict the effects of certain parameter changes.

In the first section of this chapter, we develop conditions under which the optimal basis remains the same despite a change in the problem data, and we examine the consequences on the optimal cost. We also discuss how to obtain an optimal solution if we add or delete some constraints. In subsequent sections, we allow larger changes in the problem data, resulting in a new optimal basis, and we develop a global perspective of the dependence of the optimal cost on the vectors $\mathbf{b}$ and $\mathbf{c}$. The chapter ends with a brief discussion of parametric programming, which is an extension of the simplex method tailored to the case where there is a single scalar unknown parameter.

Many of the results in this chapter can be extended to cover general linear programming problems. Nevertheless, and in order to simplify the presentation, our standing assumption throughout this chapter will be that we are dealing with a standard form problem and that the rows of the $m \times n$ matrix $\mathbf{A}$ are linearly independent.

5.1 Local sensitivity analysis

In this section, we develop a methodology for performing sensitivity analysis. We consider a linear programming problem, and we assume that we already have an optimal basis $\mathbf{B}$ and the associated optimal solution $\mathbf{x}^*$. We then assume that some entry of $\mathbf{A}$, $\mathbf{b}$, or $\mathbf{c}$ has been changed, or that a new constraint is added, or that a new variable is added. We first look for conditions under which the current basis is still optimal. If these conditions are violated, we look for an algorithm that finds a new optimal solution without having to solve the new problem from scratch. We will see that the simplex method can be quite useful in this respect.

Having assumed that $\mathbf{B}$ is an optimal basis for the original problem, the following two conditions are satisfied:

$$\mathbf{B}^{-1}\mathbf{b} \geq \mathbf{0}, \qquad \text{(feasibility)}$$

$$\mathbf{c}' - \mathbf{c}_B'\mathbf{B}^{-1}\mathbf{A} \geq \mathbf{0}', \qquad \text{(optimality)}.$$

When the problem is changed, we check to see how these conditions are affected. By insisting that both conditions (feasibility and optimality) hold for the modified problem, we obtain the conditions under which the basis matrix $\mathbf{B}$ remains optimal for the modified problem. In what follows, we apply this approach to several examples.

A new variable is added

Suppose that we introduce a new variable x_{n+1}, together with a corresponding column $\mathbf{A}_{n+1}$, and obtain the new problem

$$\begin{array}{ll} \text{minimize} & \mathbf{c}'\mathbf{x} + c_{n+1}x_{n+1} \\ \text{subject to} & \mathbf{A}\mathbf{x} + \mathbf{A}_{n+1}x_{n+1} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}. \end{array}$$

We wish to determine whether the current basis $\mathbf{B}$ is still optimal.

We note that $(\mathbf{x}, x_{n+1}) = (\mathbf{x}^*, 0)$ is a basic feasible solution to the new problem associated with the basis $\mathbf{B}$, and we only need to examine the optimality conditions. For the basis $\mathbf{B}$ to remain optimal, it is necessary and sufficient that the reduced cost of x_{n+1} be nonnegative, that is,

$$\bar{c}_{n+1} = c_{n+1} - \mathbf{c}_B'\mathbf{B}^{-1}\mathbf{A}_{n+1} \geq 0.$$

If this condition is satisfied, $(\mathbf{x}^*, 0)$ is an optimal solution to the new problem. If, however, $\bar{c}_{n+1} < 0$, then $(\mathbf{x}^*, 0)$ is not necessarily optimal. In order to find an optimal solution, we add a column to the simplex tableau, associated with the new variable, and apply the primal simplex algorithm starting from the current basis $\mathbf{B}$. Typically, an optimal solution to the new problem is obtained with a small number of iterations, and this approach is usually much faster than solving the new problem from scratch.

Example 5.1 Consider the problem

$$\begin{array}{lrcrcrcrcl} \text{minimize} & -5x_1 & - & x_2 & + & 12x_3 & & & & \\ \text{subject to} & 3x_1 & + & 2x_2 & + & x_3 & & & = & 10 \\ & 5x_1 & + & 3x_2 & & & + & x_4 & = & 16 \\ & \multicolumn{9}{l}{x_1, \ldots, x_4 \geq 0.} \end{array}$$

An optimal solution to this problem is given by $\mathbf{x} = (2, 2, 0, 0)$ and the corresponding simplex tableau is given by

		x_1	x_2	x_3	x_4
	12	0	0	2	7
$x_1 =$	2	1	0	−3	2
$x_2 =$	2	0	1	5	−3

Note that $\mathbf{B}^{-1}$ is given by the last two columns of the tableau.

Let us now introduce a variable x_5 and consider the new problem

$$\begin{array}{lrcrcrcrcrcl} \text{minimize} & -5x_1 & - & x_2 & + & 12x_3 & & & - & x_5 & & \\ \text{subject to} & 3x_1 & + & 2x_2 & + & x_3 & & & + & x_5 & = & 10 \\ & 5x_1 & + & 3x_2 & & & + & x_4 & + & x_5 & = & 16 \\ & & & x_1, \ldots, x_5 \geq 0. & & & & & & & & \end{array}$$

We have $\mathbf{A}_5 = (1, 1)$ and

$$\bar{c}_5 = c_5 - \mathbf{c}_B'\mathbf{B}^{-1}\mathbf{A}_5 = -1 - \begin{bmatrix} -5 & -1 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -4.$$

Since $\bar{c}_5$ is negative, introducing the new variable to the basis can be beneficial. We observe that $\mathbf{B}^{-1}\mathbf{A}_5 = (-1, 2)$ and augment the tableau by introducing a column associated with x_5:

		x_1	x_2	x_3	x_4	x_5
	12	0	0	2	7	−4
$x_1 =$	2	1	0	−3	2	−1
$x_2 =$	2	0	1	5	−3	2

We then bring x_5 into the basis; x_2 exits and we obtain the following tableau, which happens to be optimal:

		x_1	x_2	x_3	x_4	x_5
	16	0	2	12	1	0
$x_1 =$	3	1	0.5	−0.5	0.5	0
$x_5 =$	1	0	0.5	2.5	−1.5	1

An optimal solution is given by $\mathbf{x} = (3, 0, 0, 0, 1)$.

A new inequality constraint is added

Let us now introduce a new constraint $\mathbf{a}_{m+1}'\mathbf{x} \geq b_{m+1}$, where $\mathbf{a}_{m+1}$ and b_{m+1} are given. If the optimal solution $\mathbf{x}^*$ to the original problem satisfies this constraint, then $\mathbf{x}^*$ is an optimal solution to the new problem as well. If the new constraint is violated, we introduce a nonnegative slack variable x_{n+1}, and rewrite the new constraint in the form $\mathbf{a}_{m+1}'\mathbf{x} - x_{n+1} = b_{m+1}$. We obtain a problem in standard form, in which the matrix $\mathbf{A}$ is replaced by

$$\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{a}_{m+1}' & -1 \end{bmatrix}.$$

Let $\mathbf{B}$ be an optimal basis for the original problem. We form a basis for the new problem by selecting the original basic variables together with x_{n+1}. The new basis matrix $\overline{\mathbf{B}}$ is of the form

$$\overline{\mathbf{B}} = \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{a}' & -1 \end{bmatrix},$$

where the row vector $\mathbf{a}'$ contains those components of $\mathbf{a}_{m+1}'$ associated with the original basic columns. (The determinant of this matrix is the negative of the determinant of $\mathbf{B}$, hence nonzero, and we therefore have a true basis matrix.) The basic solution associated with this basis is $(\mathbf{x}^*, \mathbf{a}_{m+1}'\mathbf{x}^* - b_{m+1})$, and is infeasible because of our assumption that $\mathbf{x}^*$ violates the new constraint. Note that the new inverse basis matrix is readily available because

$$\overline{\mathbf{B}}^{-1} = \begin{bmatrix} \mathbf{B}^{-1} & \mathbf{0} \\ \mathbf{a}'\mathbf{B}^{-1} & -1 \end{bmatrix}.$$

(To see this, note that the product $\overline{\mathbf{B}}^{-1}\overline{\mathbf{B}}$ is equal to the identity matrix.)

Let $\mathbf{c}_B$ be the m-dimensional vector with the costs of the basic variables in the original problem. Then, the vector of reduced costs associated with the basis $\overline{\mathbf{B}}$ for the new problem, is given by

$$\begin{bmatrix} \mathbf{c}' & 0 \end{bmatrix} - \begin{bmatrix} \mathbf{c}_B' & 0 \end{bmatrix} \begin{bmatrix} \mathbf{B}^{-1} & \mathbf{0} \\ \mathbf{a}'\mathbf{B}^{-1} & -1 \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{a}_{m+1}' & -1 \end{bmatrix} = \begin{bmatrix} \mathbf{c}' - \mathbf{c}_B'\mathbf{B}^{-1}\mathbf{A} & 0 \end{bmatrix},$$

and is nonnegative due to the optimality of $\mathbf{B}$ for the original problem. Hence, $\overline{\mathbf{B}}$ is a dual feasible basis and we are in a position to apply the dual simplex method to the new problem. Note that an initial simplex tableau for the new problem is readily constructed. For example, we have

$$\overline{\mathbf{B}}^{-1} \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{a}_{m+1}' & -1 \end{bmatrix} = \begin{bmatrix} \mathbf{B}^{-1}\mathbf{A} & \mathbf{0} \\ \mathbf{a}'\mathbf{B}^{-1}\mathbf{A} - \mathbf{a}_{m+1}' & 1 \end{bmatrix},$$

where $\mathbf{B}^{-1}\mathbf{A}$ is available from the final simplex tableau for the original problem.

Example 5.2 Consider again the problem in Example 5.1:

$$\begin{array}{lrcrcrcrcl} \text{minimize} & -5x_1 & - & x_2 & + & 12x_3 & & & & \\ \text{subject to} & 3x_1 & + & 2x_2 & + & x_3 & & & = & 10 \\ & 5x_1 & + & 3x_2 & & & + & x_4 & = & 16 \\ & & & x_1, \ldots, x_4 \geq 0, & & & & & & \end{array}$$

and recall the optimal simplex tableau:

		x_1	x_2	x_3	x_4
	12	0	0	2	7
$x_1 =$	2	1	0	−3	2
$x_2 =$	2	0	1	5	−3

We introduce the additional constraint $x_1 + x_2 \geq 5$, which is violated by the optimal solution $\mathbf{x}^* = (2,2,0,0)$. We have $\mathbf{a}_{m+1} = (1,1,0,0)$, $b_{m+1} = 5$, and $\mathbf{a}'_{m+1}\mathbf{x}^* < b_{m+1}$. We form the standard form problem

$$\begin{array}{llllllll}
\text{minimize} & -5x_1 & - & x_2 & + & 12x_3 & & \\
\text{subject to} & 3x_1 & + & 2x_2 & + & x_3 & & = 10 \\
& 5x_1 & + & 3x_2 & & & + x_4 & = 16 \\
& x_1 & + & x_2 & & & - x_5 & = 5 \\
& x_1, \ldots, x_5 \geq 0. & & & & & &
\end{array}$$

Let $\mathbf{a}$ consist of the components of $\mathbf{a}_{m+1}$ associated with the basic variables. We then have $\mathbf{a} = (1,1)$ and

$$\mathbf{a}'\mathbf{B}^{-1}\mathbf{A} - \mathbf{a}'_{m+1} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & 5 & -3 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 & -1 \end{bmatrix}.$$

The tableau for the new problem is of the form

		x_1	x_2	x_3	x_4	x_5
	12	0	0	2	7	0
$x_1 =$	2	1	0	−3	2	0
$x_2 =$	2	0	1	5	−3	0
$x_5 =$	−1	0	0	2	−1	1

We now have all the information necessary to apply the dual simplex method to the new problem.

Our discussion has been focused on the case where an inequality constraint is added to the primal problem. Suppose now that we introduce a new constraint $\mathbf{p}'\mathbf{A}_{n+1} \leq c_{n+1}$ in the dual. This is equivalent to introducing a new variable in the primal, and we are back to the case that was considered in the preceding subsection.

A new equality constraint is added

We now consider the case where the new constraint is of the form $\mathbf{a}'_{m+1}\mathbf{x} = b_{m+1}$, and we assume that this new constraint is violated by the optimal solution $\mathbf{x}^*$ to the original problem. The dual of the new problem is

$$\begin{array}{ll}
\text{maximize} & \mathbf{p}'\mathbf{b} + p_{m+1}b_{m+1} \\
\text{subject to} & \begin{bmatrix} \mathbf{p}' & p_{m+1} \end{bmatrix} \begin{bmatrix} \mathbf{A} \\ \mathbf{a}'_{m+1} \end{bmatrix} \leq \mathbf{c}',
\end{array}$$

where p_{m+1} is a dual variable associated with the new constraint. Let $\mathbf{p}^*$ be an optimal basic feasible solution to the original dual problem. Then, $(\mathbf{p}^*, 0)$ is a feasible solution to the new dual problem.

Let m be the dimension of $\mathbf{p}$, which is the same as the original number of constraints. Since $\mathbf{p}^*$ is a basic feasible solution to the original dual problem, m of the constraints in $(\mathbf{p}^*)'\mathbf{A} \leq \mathbf{c}'$ are linearly independent and active. However, there is no guarantee that at $(\mathbf{p}^*, 0)$ we will have $m+1$ linearly independent active constraints of the new dual problem. In particular, $(\mathbf{p}^*, 0)$ need not be a basic feasible solution to the new dual problem and may not provide a convenient starting point for the dual simplex method on the new problem. While it may be possible to obtain a dual basic feasible solution by setting p_{m+1} to a suitably chosen nonzero value, we present here an alternative approach.

Let us assume, without loss of generality, that $\mathbf{a}'_{m+1}\mathbf{x}^* > b_{m+1}$. We introduce the auxiliary primal problem

$$\begin{array}{llll}
\text{minimize} & \mathbf{c}'\mathbf{x} & + \; Mx_{n+1} & \\
\text{subject to} & \mathbf{A}\mathbf{x} & & = \mathbf{b} \\
& \mathbf{a}'_{m+1}\mathbf{x} & - \; x_{n+1} & = b_{m+1} \\
& \mathbf{x} \geq \mathbf{0}, & x_{n+1} \geq 0, &
\end{array}$$

where M is a large positive constant. A primal feasible basis for the auxiliary problem is obtained by picking the basic variables of the optimal solution to the original problem, together with the variable x_{n+1}. The resulting basis matrix is the same as the matrix $\overline{\mathbf{B}}$ of the preceding subsection. There is a difference, however. In the preceding subsection, $\overline{\mathbf{B}}$ was a dual feasible basis, whereas here $\overline{\mathbf{B}}$ is a primal feasible basis. For this reason, the primal simplex method can now be used to solve the auxiliary problem to optimality.

Suppose that an optimal solution to the auxiliary problem satisfies $x_{n+1} = 0$; this will be the case if the new problem is feasible and the coefficient M is large enough. Then, the additional constraint $\mathbf{a}'_{m+1}\mathbf{x} = b_{m+1}$ has been satisfied and we have an optimal solution to the new problem.

Changes in the requirement vector b

Suppose that some component b_i of the requirement vector $\mathbf{b}$ is changed to $b_i + \delta$. Equivalently, the vector $\mathbf{b}$ is changed to $\mathbf{b} + \delta\mathbf{e}_i$, where $\mathbf{e}_i$ is the ith unit vector. We wish to determine the range of values of δ under which the current basis remains optimal. Note that the optimality conditions are not affected by the change in $\mathbf{b}$. We therefore need to examine only the feasibility condition

$$\mathbf{B}^{-1}(\mathbf{b} + \delta\mathbf{e}_i) \geq \mathbf{0}. \qquad (5.1)$$

Let $\mathbf{g} = (\beta_{1i}, \beta_{2i}, \ldots, \beta_{mi})$ be the ith column of $\mathbf{B}^{-1}$. Equation (5.1) becomes

$$\mathbf{x}_B + \delta\mathbf{g} \geq \mathbf{0},$$

or,

$$x_{B(j)} + \delta\beta_{ji} \geq 0, \qquad j = 1, \ldots, m.$$

Equivalently,

$$\max_{\{j|\beta_{ji}>0\}} \left(-\frac{x_{B(j)}}{\beta_{ji}}\right) \le \delta \le \min_{\{j|\beta_{ji}<0\}} \left(-\frac{x_{B(j)}}{\beta_{ji}}\right).$$

For δ in this range, the optimal cost, as a function of δ, is given by $\mathbf{c}_B'\mathbf{B}^{-1}(\mathbf{b}+\delta\mathbf{e}_i) = \mathbf{p}'\mathbf{b} + \delta p_i$, where $\mathbf{p}' = \mathbf{c}_B'\mathbf{B}^{-1}$ is the (optimal) dual solution associated with the current basis $\mathbf{B}$.

If δ is outside the allowed range, the current solution satisfies the optimality (or dual feasibility) conditions, but is primal infeasible. In that case, we can apply the dual simplex algorithm starting from the current basis.

Example 5.3 Consider the optimal tableau

		x_1	x_2	x_3	x_4
	12	0	0	2	7
$x_1 =$	2	1	0	−3	2
$x_2 =$	2	0	1	5	−3

from Example 5.1.

Let us contemplate adding δ to b_1. We look at the first column of $\mathbf{B}^{-1}$ which is $(-3, 5)$. The basic variables under the same basis are $x_1 = 2 - 3\delta$ and $2 + 5\delta$. This basis will remain feasible as long as $2 - 3\delta \ge 0$ and $2 + 5\delta \ge 0$, that is, if $-2/5 \le \delta \le 2/3$. The rate of change of the optimal cost per unit change of δ is given by $\mathbf{c}_B'\mathbf{B}^{-1}\mathbf{e}_1 = (-5, -1)'(-3, 5) = 10$.

If δ is increased beyond 2/3, then x_1 becomes negative. At this point, we can perform an iteration of the dual simplex method to remove x_1 from the basis, and x_3 enters the basis.

Changes in the cost vector c

Suppose now that some cost coefficient c_j becomes $c_j + \delta$. The primal feasibility condition is not affected. We therefore need to focus on the optimality condition

$$\mathbf{c}_B'\mathbf{B}^{-1}\mathbf{A} \le \mathbf{c}'.$$

If c_j is the cost coefficient of a nonbasic variable x_j, then $\mathbf{c}_B$ does not change, and the only inequality that is affected is the one for the reduced cost of x_j; we need

$$\mathbf{c}_B'\mathbf{B}^{-1}\mathbf{A}_j \le c_j + \delta,$$

or

$$\delta \ge -\bar{c}_j.$$

If this condition holds, the current basis remains optimal; otherwise, we can apply the primal simplex method starting from the current basic feasible solution.

If c_j is the cost coefficient of the ℓth basic variable, that is, if $j = B(\ell)$, then $\mathbf{c}_B$ becomes $\mathbf{c}_B + \delta\mathbf{e}_\ell$ and all of the optimality conditions will be affected. The optimality conditions for the new problem are

$$(\mathbf{c}_B + \delta\mathbf{e}_\ell)'\mathbf{B}^{-1}\mathbf{A}_i \le c_i, \qquad \forall\, i \ne j.$$

(Since x_j is a basic variable, its reduced cost stays at zero and need not be examined.) Equivalently,

$$\delta q_{\ell i} \le \bar{c}_i, \qquad \forall\, i \ne j,$$

where $q_{\ell i}$ is the ℓth entry of $\mathbf{B}^{-1}\mathbf{A}_i$, which can be obtained from the simplex tableau. These inequalities determine the range of δ for which the same basis remains optimal.

Example 5.4 We consider once more the problem in Example 5.1 and determine the range of changes δ_i of c_i, under which the same basis remains optimal. Since x_3 and x_4 are nonbasic variables, we obtain the conditions

$$\delta_3 \ge -\bar{c}_3 = -2,$$
$$\delta_4 \ge -\bar{c}_4 = -7.$$

Consider now adding δ_1 to c_1. From the simplex tableau, we obtain $q_{12} = 0$, $q_{13} = -3$, $q_{14} = 2$, and we are led to the conditions

$$\delta_1 \ge -2/3,$$
$$\delta_1 \le 7/2.$$

Changes in a nonbasic column of A

Suppose that some entry a_{ij} in the jth column $\mathbf{A}_j$ of the matrix $\mathbf{A}$ is changed to $a_{ij} + \delta$. We wish to determine the range of values of δ for which the old optimal basis remains optimal.

If the column $\mathbf{A}_j$ is nonbasic, the basis matrix $\mathbf{B}$ does not change, and the primal feasibility condition is unaffected. Furthermore, only the reduced cost of the jth column is affected, leading to the condition

$$c_j - \mathbf{p}'(\mathbf{A}_j + \delta\mathbf{e}_i) \ge 0,$$

or,

$$\bar{c}_j - \delta p_i \ge 0,$$

where $\mathbf{p}' = \mathbf{c}_B'\mathbf{B}^{-1}$. If this condition is violated, the nonbasic column $\mathbf{A}_j$ can be brought into the basis, and we can continue with the primal simplex method.

Changes in a basic column of A

If one of the entries of a basic column $\mathbf{A}_j$ changes, then both the feasibility and optimality conditions are affected. This case is more complicated and we leave the full development for the exercises. As it turns out, the range of values of δ for which the same basis is optimal is again an interval (Exercise 5.3).

Suppose that the basic column $\mathbf{A}_j$ is changed to $\mathbf{A}_j + \delta \mathbf{e}_i$, where $\mathbf{e}_i$ is the ith unit vector. Assume that both the original problem and its dual have unique and nondegenerate optimal solutions $\mathbf{x}^*$ and $\mathbf{p}$, respectively. Let $\mathbf{x}^*(\delta)$ be an optimal solution to the modified problem, as a function of δ. It can be shown (Exercise 5.2) that for small δ we have

$$\mathbf{c}'\mathbf{x}^*(\delta) = \mathbf{c}'\mathbf{x}^* - \delta x_j^* p_i + O(\delta^2).$$

For an intuitive interpretation of this equation, let us consider the diet problem and recall that a_{ij} corresponds to the amount of the ith nutrient in the jth food. Given an optimal solution $\mathbf{x}^*$ to the original problem, an increase of a_{ij} by δ means that we are getting "for free" an additional amount δx_j^* of the ith nutrient. Since the dual variable p_i is the marginal cost per unit of the ith nutrient, we are getting for free something that is normally worth $\delta p_i x_j^*$, and this allows us to reduce our costs by that same amount.

Production planning revisited

In Section 1.2, we introduced a production planning problem that DEC had faced in the end of 1988. In this section, we answer some of the questions that we posed. Recall that there were two important choices, whether to use the constrained or the unconstrained mode of production for disk drives, and whether to use alternative memory boards. As discussed in Section 1.2, these four combinations of choices led to four different linear programming problems. We report the solution to these problems, as obtained from a linear programming package, in Table 5.1.

Table 5.1 indicates that revenues can substantially increase by using alternative memory boards, and the company should definitely do so. The decision of whether to use the constrained or the unconstrained mode of production for disk drives is less clear. In the constrained mode, the revenue is 248 million versus 213 million in the unconstrained mode. However, customer satisfaction and, therefore, future revenues might be affected, since in the constrained mode some customers will get a product different than the desired one. Moreover, these results are obtained assuming that the number of available 256K memory boards and disk drives were 8,000 and 3,000, respectively, which is the lowest value in the range that was estimated. We should therefore examine the sensitivity of the solution as the number of available 256K memory boards and disk drives increases.

Alt. boards	Mode	Revenue	x_1	x_2	x_3	x_4	x_5
no	constr.	145	0	2.5	0	0.5	2
yes	constr.	248	1.8	2	0	1	2
no	unconstr.	133	0.272	1.304	0.3	0.5	2.7
yes	unconstr.	213	1.8	1.035	0.3	0.5	2.7

Table 5.1: Optimal solutions to the four variants of the production planning problem. Revenue is in millions of dollars and the quantities x_i are in thousands.

With most linear programming packages, the output includes the values of the dual variables, as well as the range of parameter variations under which local sensitivity analysis is valid. Table 5.2 presents the values of the dual variables associated with the constraints on available disk drives and 256K memory boards. In addition, it provides the range of allowed changes on the number of disk drives and memory boards that would leave the dual variables unchanged. This information is provided for the two linear programming problems corresponding to constrained and unconstrained mode of production for disk drives, respectively, under the assumption that alternative memory boards will be used.

Mode	Constrained	Unconstrained
Revenue	248	213
Dual variable for 256K boards	15	0
Range for 256K boards	$[-1.5, 0.2]$	$[-1.62, \infty]$
Dual variable for disk drives	0	23.52
Range for disk drives	$[-0.2, 0.75]$	$[-0.91, 1.13]$

Table 5.2: Dual prices and ranges for the constraints corresponding to the availability of the number of 256K memory boards and disk drives.

In the constrained mode, increasing the number of available 256K boards by 0.2 thousand (the largest number in the allowed range) results in a revenue increase of $15 \times 0.2 = 3$ million. In the unconstrained mode, increasing the number of available 256K boards has no effect on revenues, because the dual variable is zero and the range extends upwards to infinity. In the constrained mode, increasing the number of available disk drives by up to 0.75 thousand (the largest number in the allowed range) has no effect on revenue. Finally, in the unconstrained mode, increasing the number of available disk drives by 1.13 thousand results in a revenue increase of $23.52 \times 1.13 = 26.57$ million.

In conclusion, in the constrained mode of production, it is important to aim at an increase of the number of available 256K memory boards, while in the unconstrained mode, increasing the number of disk drives is more important.

This example demonstrates that even a small linear programming problem (with five variables, in this case) can have an impact on a company's planning process. Moreover, the information provided by linear programming solvers (dual variables, ranges, etc.) can offer significant insights and can be a very useful aid to decision makers.

5.2 Global dependence on the right-hand side vector

In this section, we take a global view of the dependence of the optimal cost on the requirement vector $\mathbf{b}$.

Let
$$P(\mathbf{b}) = \{\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$$

be the feasible set, and note that our notation makes the dependence on $\mathbf{b}$ explicit. Let
$$S = \{\mathbf{b} \mid P(\mathbf{b}) \text{ is nonempty}\},$$

and observe that
$$S = \{\mathbf{A}\mathbf{x} \mid \mathbf{x} \geq \mathbf{0}\};$$

in particular, S is a convex set. For any $\mathbf{b} \in S$, we define
$$F(\mathbf{b}) = \min_{\mathbf{x} \in P(\mathbf{b})} \mathbf{c}'\mathbf{x},$$

which is the optimal cost as a function of $\mathbf{b}$.

Throughout this section, we assume that the dual feasible set $\{\mathbf{p} \mid \mathbf{p}'\mathbf{A} \leq \mathbf{c}'\}$ is nonempty. Then, duality theory implies that the optimal primal cost $F(\mathbf{b})$ is finite for every $\mathbf{b} \in S$. Our goal is to understand the structure of the function $F(\mathbf{b})$, for $\mathbf{b} \in S$.

Let us fix a particular element $\mathbf{b}^*$ of S. Suppose that there exists a nondegenerate primal optimal basic feasible solution, and let $\mathbf{B}$ be the corresponding optimal basis matrix. The vector $\mathbf{x}_B$ of basic variables at that optimal solution is given by $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}^*$, and is positive by nondegeneracy. In addition, the vector of reduced costs is nonnegative. If we change $\mathbf{b}^*$ to $\mathbf{b}$ and if the difference $\mathbf{b} - \mathbf{b}^*$ is sufficiently small, $\mathbf{B}^{-1}\mathbf{b}$ remains positive and we still have a basic feasible solution. The reduced costs are not affected by the change from $\mathbf{b}^*$ to $\mathbf{b}$ and remain nonnegative. Therefore, $\mathbf{B}$ is an optimal basis for the new problem as well. The optimal cost $F(\mathbf{b})$ for the new problem is given by
$$F(\mathbf{b}) = \mathbf{c}_B'\mathbf{B}^{-1}\mathbf{b} = \mathbf{p}'\mathbf{b}, \qquad \text{for } \mathbf{b} \text{ close to } \mathbf{b}^*,$$

where $\mathbf{p}' = \mathbf{c}_B'\mathbf{B}^{-1}$ is the optimal solution to the dual problem. This establishes that in the vicinity of $\mathbf{b}^*$, $F(\mathbf{b})$ is a linear function of $\mathbf{b}$ and its gradient is given by $\mathbf{p}$.

We now turn to the global properties of $F(\mathbf{b})$.

Theorem 5.1 *The optimal cost $F(\mathbf{b})$ is a convex function of $\mathbf{b}$ on the set S.*

Proof. Let $\mathbf{b}^1$ and $\mathbf{b}^2$ be two elements of S. For $i = 1, 2$, let $\mathbf{x}^i$ be an optimal solution to the problem of minimizing $\mathbf{c}'\mathbf{x}$ subject to $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{A}\mathbf{x} = \mathbf{b}^i$. Thus, $F(\mathbf{b}^1) = \mathbf{c}'\mathbf{x}^1$ and $F(\mathbf{b}^2) = \mathbf{c}'\mathbf{x}^2$. Fix a scalar $\lambda \in [0, 1]$, and note that the vector $\mathbf{y} = \lambda\mathbf{x}^1 + (1-\lambda)\mathbf{x}^2$ is nonnegative and satisfies $\mathbf{A}\mathbf{y} = \lambda\mathbf{b}^1 + (1-\lambda)\mathbf{b}^2$. In particular, $\mathbf{y}$ is a feasible solution to the linear programming problem obtained when the requirement vector $\mathbf{b}$ is set to $\lambda\mathbf{b}^1 + (1-\lambda)\mathbf{b}^2$. Therefore,
$$F\big(\lambda\mathbf{b}^1 + (1-\lambda)\mathbf{b}^2\big) \leq \mathbf{c}'\mathbf{y} = \lambda\mathbf{c}'\mathbf{x}^1 + (1-\lambda)\mathbf{c}'\mathbf{x}^2 = \lambda F(\mathbf{b}^1) + (1-\lambda)F(\mathbf{b}^2),$$

establishing the convexity of F. □

We now corroborate Theorem 5.1 by taking a different approach, involving the dual problem
$$\begin{array}{ll} \text{maximize} & \mathbf{p}'\mathbf{b} \\ \text{subject to} & \mathbf{p}'\mathbf{A} \leq \mathbf{c}', \end{array}$$

which has been assumed feasible. For any $\mathbf{b} \in S$, $F(\mathbf{b})$ is finite and, by strong duality, is equal to the optimal value of the dual objective. Let $\mathbf{p}^1, \mathbf{p}^2, \ldots, \mathbf{p}^N$ be the extreme points of the dual feasible set. (Our standing assumption is that the matrix $\mathbf{A}$ has linearly independent rows; hence its columns span $\Re^m$. Equivalently, the rows of $\mathbf{A}'$ span $\Re^m$ and Theorem 2.6 in Section 2.5 implies that the dual feasible set must have at least one

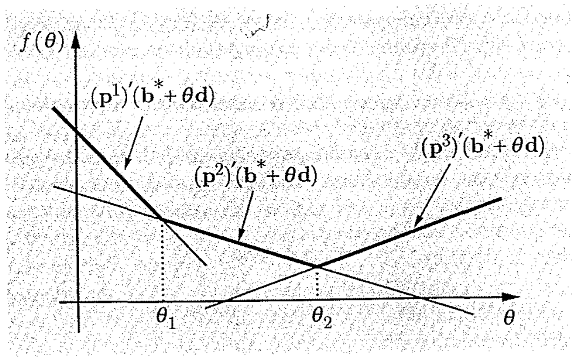

Figure 5.1: The optimal cost when the vector $\mathbf{b}$ is a function of a scalar parameter. Each linear piece is of the form $(\mathbf{p}^i)'(\mathbf{b}^* + \theta\mathbf{d})$, where $\mathbf{p}^i$ is the ith extreme point of the dual feasible set. In each one of the intervals $\theta < \theta_1$, $\theta_1 < \theta < \theta_2$, and $\theta > \theta_2$, we have different dual optimal solutions, namely, $\mathbf{p}^1$, $\mathbf{p}^2$, and $\mathbf{p}^3$, respectively. For $\theta = \theta_1$ or $\theta = \theta_2$, the dual problem has multiple optimal solutions.

extreme point.) Since the optimum of the dual must be attained at an extreme point, we obtain

$$F(\mathbf{b}) = \max_{i=1,\ldots,N} (\mathbf{p}^i)'\mathbf{b}, \qquad \mathbf{b} \in S. \tag{5.2}$$

In particular, F is equal to the maximum of a finite collection of linear functions. It is therefore a piecewise linear convex function, and we have a new proof of Theorem 5.1. In addition, within a region where F is linear, we have $F(\mathbf{b}) = (\mathbf{p}^i)'\mathbf{b}$, where $\mathbf{p}^i$ is a corresponding dual optimal solution, in agreement with our earlier discussion.

For those values of $\mathbf{b}$ for which F is not differentiable, that is, at the junction of two or more linear pieces, the dual problem does not have a unique optimal solution and this implies that every optimal basic feasible solution to the primal is degenerate. (This is because, as shown earlier in this section, the existence of a nondegenerate optimal basic feasible solution to the primal implies that F is locally linear.)

We now restrict attention to changes in $\mathbf{b}$ of a particular type, namely, $\mathbf{b} = \mathbf{b}^* + \theta\mathbf{d}$, where $\mathbf{b}^*$ and $\mathbf{d}$ are fixed vectors and θ is a scalar. Let $f(\theta) = F(\mathbf{b}^* + \theta\mathbf{d})$ be the optimal cost as a function of the scalar parameter θ. Using Eq. (5.2), we obtain

$$f(\theta) = \max_{i=1,\ldots,N} (\mathbf{p}^i)'(\mathbf{b}^* + \theta\mathbf{d}), \qquad \mathbf{b}^* + \theta\mathbf{d} \in S.$$

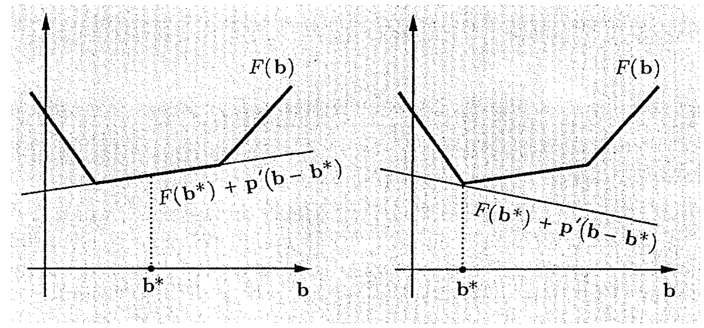

Figure 5.2: Illustration of subgradients of a function F at a point $\mathbf{b}^*$. A subgradient $\mathbf{p}$ is the gradient of a linear function $F(\mathbf{b}^*) + \mathbf{p}'(\mathbf{b} - \mathbf{b}^*)$ that lies below the function $F(\mathbf{b})$ and agrees with it for $\mathbf{b} = \mathbf{b}^*$.

This is essentially a "section" of the function F; it is again a piecewise linear convex function; see Figure 5.1. Once more, at breakpoints of this function, every optimal basic feasible solution to the primal must be degenerate.

5.3 The set of all dual optimal solutions*

We have seen that if the function F is defined, finite, and linear in the vicinity of a certain vector $\mathbf{b}^*$, then there is a unique optimal dual solution, equal to the gradient of F at that point, which leads to the interpretation of dual optimal solutions as marginal costs. We would like to extend this interpretation so that it remains valid at the breakpoints of F. This is indeed possible: we will show shortly that any dual optimal solution can be viewed as a "generalized gradient" of F. We first need the following definition, which is illustrated in Figure 5.2.

Definition 5.1 *Let F be a convex function defined on a convex set S. Let $\mathbf{b}^*$ be an element of S. We say that a vector $\mathbf{p}$ is a* **subgradient** *of F at $\mathbf{b}^*$ if*

$$F(\mathbf{b}^*) + \mathbf{p}'(\mathbf{b} - \mathbf{b}^*) \le F(\mathbf{b}), \qquad \forall\, \mathbf{b} \in S.$$

Note that if $\mathbf{b}^*$ is a breakpoint of the function F, then there are several subgradients. On the other hand, if F is linear near $\mathbf{b}^*$, there is a unique subgradient, equal to the gradient of F.

Theorem 5.2 *Suppose that the linear programming problem of minimizing* $\mathbf{c}'\mathbf{x}$ *subject to* $\mathbf{Ax} = \mathbf{b}^*$ *and* $\mathbf{x} \geq \mathbf{0}$ *is feasible and that the optimal cost is finite. Then, a vector* $\mathbf{p}$ *is an optimal solution to the dual problem if and only if it is a subgradient of the optimal cost function* F *at the point* $\mathbf{b}^*$.

Proof. Recall that the function F is defined on the set S, which is the set of vectors $\mathbf{b}$ for which the set $P(\mathbf{b})$ of feasible solutions to the primal problem is nonempty. Suppose that $\mathbf{p}$ is an optimal solution to the dual problem. Then, strong duality implies that $\mathbf{p}'\mathbf{b}^* = F(\mathbf{b}^*)$. Consider now some arbitrary $\mathbf{b} \in S$. For any feasible solution $\mathbf{x} \in P(\mathbf{b})$, weak duality yields $\mathbf{p}'\mathbf{b} \leq \mathbf{c}'\mathbf{x}$. Taking the minimum over all $\mathbf{x} \in P(\mathbf{b})$, we obtain $\mathbf{p}'\mathbf{b} \leq F(\mathbf{b})$. Hence, $\mathbf{p}'\mathbf{b} - \mathbf{p}'\mathbf{b}^* \leq F(\mathbf{b}) - F(\mathbf{b}^*)$, and we conclude that $\mathbf{p}$ is a subgradient of F at $\mathbf{b}^*$.

We now prove the converse. Let $\mathbf{p}$ be a subgradient of F at $\mathbf{b}^*$; that is,

$$F(\mathbf{b}^*) + \mathbf{p}'(\mathbf{b} - \mathbf{b}^*) \leq F(\mathbf{b}), \qquad \forall\ \mathbf{b} \in S. \tag{5.3}$$

Pick some $\mathbf{x} \geq \mathbf{0}$, let $\mathbf{b} = \mathbf{Ax}$, and note that $\mathbf{x} \in P(\mathbf{b})$. In particular, $F(\mathbf{b}) \leq \mathbf{c}'\mathbf{x}$. Using Eq. (5.3), we obtain

$$\mathbf{p}'\mathbf{Ax} = \mathbf{p}'\mathbf{b} \leq F(\mathbf{b}) - F(\mathbf{b}^*) + \mathbf{p}'\mathbf{b}^* \leq \mathbf{c}'\mathbf{x} - F(\mathbf{b}^*) + \mathbf{p}'\mathbf{b}^*.$$

Since this is true for all $\mathbf{x} \geq \mathbf{0}$, we must have $\mathbf{p}'\mathbf{A} \leq \mathbf{c}'$, which shows that $\mathbf{p}$ is a dual feasible solution. Also, by letting $\mathbf{x} = \mathbf{0}$, we obtain $F(\mathbf{b}^*) \leq \mathbf{p}'\mathbf{b}^*$. Using weak duality, every dual feasible solution $\mathbf{q}$ must satisfy $\mathbf{q}'\mathbf{b}^* \leq F(\mathbf{b}^*) \leq \mathbf{p}'\mathbf{b}^*$, which shows that $\mathbf{p}$ is a dual optimal solution. □

5.4 Global dependence on the cost vector

In the last two sections, we fixed the matrix $\mathbf{A}$ and the vector $\mathbf{c}$, and we considered the effect of changing the vector $\mathbf{b}$. The key to our development was the fact that the set of dual feasible solutions remains the same as $\mathbf{b}$ varies. In this section, we study the case where $\mathbf{A}$ and $\mathbf{b}$ are fixed, but the vector $\mathbf{c}$ varies. In this case, the primal feasible set remains unaffected; our standing assumption will be that it is nonempty.

We define the dual feasible set

$$Q(\mathbf{c}) = \{\mathbf{p} \mid \mathbf{p}'\mathbf{A} \leq \mathbf{c}\},$$

and let

$$T = \{\mathbf{c} \mid Q(\mathbf{c}) \text{ is nonempty}\}.$$

If $\mathbf{c}^1 \in T$ and $\mathbf{c}^2 \in T$, then there exist $\mathbf{p}^1$ and $\mathbf{p}^2$ such that $(\mathbf{p}^1)'\mathbf{A} \leq \mathbf{c}'$ and $(\mathbf{p}^2)'\mathbf{A} \leq \mathbf{c}'$. For any scalar $\lambda \in [0,1]$, we have

$$\big(\lambda(\mathbf{p}^1)' + (1-\lambda)(\mathbf{p}^2)'\big)\mathbf{A} \leq \lambda\mathbf{c}^1 + (1-\lambda)\mathbf{c}^2,$$

and this establishes that $\lambda\mathbf{c}^1 + (1-\lambda)\mathbf{c}^2 \in T$. We have therefore shown that T is a convex set.

If $\mathbf{c} \notin T$, the infeasibility of the dual problem implies that the optimal primal cost is $-\infty$. On the other hand, if $\mathbf{c} \in T$, the optimal primal cost must be finite. Thus, the optimal primal cost, which we will denote by $G(\mathbf{c})$, is finite if and only if $\mathbf{c} \in T$.

Let $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N$ be the basic feasible solutions in the primal feasible set; clearly, these do not depend on $\mathbf{c}$. Since an optimal solution to a standard form problem can always be found at an extreme point, we have

$$G(\mathbf{c}) = \min_{i=1,\ldots,N} \mathbf{c}'\mathbf{x}^i.$$

Thus, $G(\mathbf{c})$ is the minimum of a finite collection of linear functions and is a piecewise linear concave function. If for some value $\mathbf{c}^*$ of $\mathbf{c}$, the primal has a unique optimal solution $\mathbf{x}^i$, we have $(\mathbf{c}^*)'\mathbf{x}^i < (\mathbf{c}^*)'\mathbf{x}^j$, for all $j \neq i$. For $\mathbf{c}$ very close to $\mathbf{c}^*$, the inequalities $\mathbf{c}'\mathbf{x}^i < \mathbf{c}'\mathbf{x}^j$, $j \neq i$, continue to hold, implying that $\mathbf{x}^i$ is still a unique primal optimal solution with cost $\mathbf{c}'\mathbf{x}^i$. We conclude that, locally, $G(\mathbf{c}) = \mathbf{c}'\mathbf{x}^i$. On the other hand, at those values of $\mathbf{c}$ that lead to multiple primal optimal solutions, the function G has a breakpoint.

We summarize the main points of the preceding discussion.

Theorem 5.3 *Consider a feasible linear programming problem in standard form.*

(a) *The set* T *of all* $\mathbf{c}$ *for which the optimal cost is finite, is convex.*

(b) *The optimal cost* $G(\mathbf{c})$ *is a concave function of* $\mathbf{c}$ *on the set* T.

(c) *If for some value of* $\mathbf{c}$ *the primal problem has a unique optimal solution* $\mathbf{x}^*$, *then* G *is linear in the vicinity of* $\mathbf{c}$ *and its gradient is equal to* $\mathbf{x}^*$.

5.5 Parametric programming

Let us fix $\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}$, and a vector $\mathbf{d}$ of the same dimension as $\mathbf{c}$. For any scalar θ, we consider the problem

$$\begin{aligned} \text{minimize} \quad & (\mathbf{c} + \theta\mathbf{d})'\mathbf{x} \\ \text{subject to} \quad & \mathbf{Ax} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}, \end{aligned}$$

and let $g(\theta)$ be the optimal cost as a function of θ. Naturally, we assume that the feasible set is nonempty. For those values of θ for which the optimal cost is finite, we have

$$g(\theta) = \min_{i=1,\ldots,N} (\mathbf{c} + \theta\mathbf{d})'\mathbf{x}^i,$$

where $\mathbf{x}^1, \ldots, \mathbf{x}^N$ are the extreme points of the feasible set; see Figure 5.3. In particular, $g(\theta)$ is a piecewise linear and concave function of the parameter θ. In this section, we discuss a systematic procedure, based on the simplex method, for obtaining $g(\theta)$ for all values of θ. We start with an example.

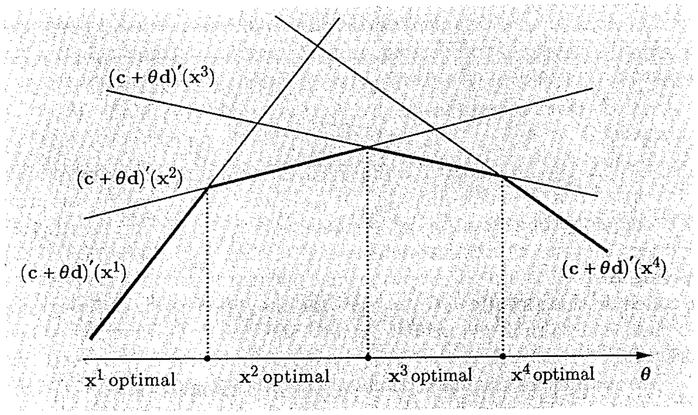

Figure 5.3: The optimal cost $g(\theta)$ as a function of θ.

Example 5.5 Consider the problem

$$\begin{array}{lrcrcrl} \text{minimize} & (-3+2\theta)x_1 & + & (3-\theta)x_2 & + & x_3 & \\ \text{subject to} & x_1 & + & 2x_2 & - & 3x_3 & \leq 5 \\ & 2x_1 & + & x_2 & - & 4x_3 & \leq 7 \\ & & & x_1, x_2, x_3 \geq 0. & & & \end{array}$$

We introduce slack variables in order to bring the problem into standard form, and then let the slack variables be the basic variables. This determines a basic feasible solution and leads to the following tableau.

		x_1	x_2	x_3	x_4	x_5
	0	$-3+2\theta$	$3-\theta$	1	0	0
$x_4 =$	5	1	2	-3	1	0
$x_5 =$	7	2	1	-4	0	1

If $-3+2\theta \geq 0$ and $3-\theta \geq 0$, all reduced costs are nonnegative and we have an optimal basic feasible solution. In particular,

$$g(\theta) = 0, \qquad \text{if } \frac{3}{2} \leq \theta \leq 3.$$

If θ is increased slightly above 3, the reduced cost of x_2 becomes negative and we no longer have an optimal basic feasible solution. We let x_2 enter the basis, x_4 exits, and we obtain the new tableau:

		x_1	x_2	x_3	x_4	x_5
	$-7.5+2.5\theta$	$-4.5+2.5\theta$	0	$5.5-1.5\theta$	$-1.5+0.5\theta$	0
$x_2 =$	2.5	0.5	1	-1.5	0.5	0
$x_5 =$	4.5	1.5	0	-2.5	-0.5	1

We note that all reduced costs are nonnegative if and only if $3 \leq \theta \leq 5.5/1.5$. The optimal cost for that range of values of θ is

$$g(\theta) = 7.5 - 2.5\theta, \qquad \text{if } 3 \leq \theta \leq \frac{5.5}{1.5}.$$

If θ is increased beyond $5.5/1.5$, the reduced cost of x_3 becomes negative. If we attempt to bring x_3 into the basis, we cannot find a positive pivot element in the third column of the tableau, and the problem is unbounded, with $g(\theta) = -\infty$.

Let us now go back to the original tableau and suppose that θ is decreased to a value slightly below $3/2$. Then, the reduced cost of x_1 becomes negative, we let x_1 enter the basis, and x_5 exits. The new tableau is:

		x_1	x_2	x_3	x_4	x_5
	$10.5-7\theta$	0	$4.5-2\theta$	$-5+4\theta$	0	$1.5-\theta$
$x_4 =$	1.5	0	1.5	-1	1	-0.5
$x_1 =$	3.5	1	0.5	-2	0	0.5

We note that all of the reduced costs are nonnegative if and only if $5/4 \leq \theta \leq 3/2$. For these values of θ, we have an optimal solution, with an optimal cost of

$$g(\theta) = -10.5 + 7\theta, \qquad \text{if } \frac{5}{4} \leq \theta \leq \frac{3}{2}.$$

Finally, for $\theta < 5/4$, the reduced cost of x_3 is negative, but the optimal cost is equal to $-\infty$, because all entries in the third column of the tableau are negative. We plot the optimal cost in Figure 5.4.

We now generalize the steps in the preceding example, in order to obtain a broader methodology. The key observation is that once a basis is fixed, the reduced costs are affine (linear plus a constant) functions of θ. Then, if we require that all reduced costs be nonnegative, we force θ to belong to some interval. (The interval could be empty but if it is nonempty, its endpoints are also included.) We conclude that for any given basis, the set of θ for which this basis is optimal is a closed interval.

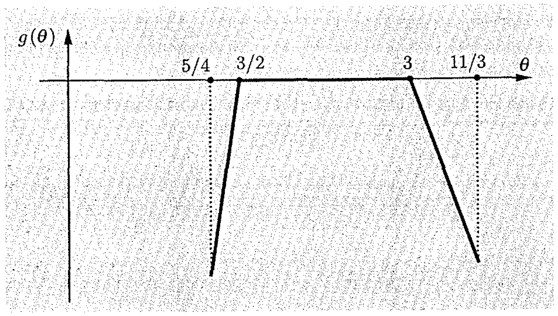

Figure 5.4: The optimal cost $g(\theta)$ as a function of θ, in Example 5.5. For θ outside the interval $[5/4, 11/3]$, $g(\theta)$ is equal to $-\infty$.

Let us now assume that we have chosen a basic feasible solution and an associated basis matrix $\mathbf{B}$, and suppose that this basis is optimal for θ satisfying $\theta_1 \leq \theta \leq \theta_2$. Let x_j be a variable whose reduced cost becomes negative for $\theta > \theta_2$. Since this reduced cost is nonnegative for $\theta_1 \leq \theta \leq \theta_2$, it must be equal to zero when $\theta = \theta_2$. We now attempt to bring x_j into the basis and consider separately the different cases that may arise.

Suppose that no entry of the jth column $\mathbf{B}^{-1}\mathbf{A}_j$ of the simplex tableau is positive. For $\theta > \theta_2$, the reduced cost of x_j is negative, and this implies that the optimal cost is $-\infty$ in that range.

If the jth column of the tableau has at least one positive element, we carry out a change of basis and obtain a new basis matrix $\overline{\mathbf{B}}$. For $\theta = \theta_2$, the reduced cost of the entering variable is zero and, therefore, the cost associated with the new basis is the same as the cost associated with the old basis. Since the old basis was optimal for $\theta = \theta_2$, the same must be true for the new basis. On the other hand, for $\theta < \theta_2$, the entering variable x_j had a positive reduced cost. According to the pivoting mechanics, and for $\theta < \theta_2$, a negative multiple of the pivot row is added to the pivot row, and this makes the reduced cost of the exiting variable negative. This implies that the new basis cannot be optimal for $\theta < \theta_2$. We conclude that the range of values of θ for which the new basis is optimal is of the form $\theta_2 \leq \theta \leq \theta_3$, for some θ_3. By continuing similarly, we obtain a sequence of bases, with the ith basis being optimal for $\theta_i \leq \theta \leq \theta_{i+1}$.

Note that a basis which is optimal for $\theta \in [\theta_i, \theta_{i+1}]$ cannot be optimal for values of θ greater than θ_{i+1}. Thus, if $\theta_{i+1} > \theta_i$ for all i, the same basis cannot be encountered more than once and the entire range of values of θ will be traced in a finite number of iterations, with each iteration leading to a new breakpoint of the optimal cost function $g(\theta)$. (The number of breakpoints may increase exponentially with the dimension of the problem.)

The situation is more complicated if for some basis we have $\theta_i = \theta_{i+1}$. In this case, it is possible that the algorithm keeps cycling between a finite number of different bases, all of which are optimal only for $\theta = \theta_i = \theta_{i+1}$. Such cycling can only happen in the presence of degeneracy in the primal problem (Exercise 5.17), but can be avoided if an appropriate anticycling rule is followed. In conclusion, the procedure we have outlined, together with an anticycling rule, partitions the range of possible values of θ into consecutive intervals and, for each interval, provides us with an optimal basis and the optimal cost function as a function of θ.

There is another variant of parametric programming that can be used when $\mathbf{c}$ is kept fixed but $\mathbf{b}$ is replaced by $\mathbf{b} + \theta\mathbf{d}$, where $\mathbf{d}$ is a given vector and θ is a scalar. In this case, the zeroth column of the tableau depends on θ. Whenever θ reaches a value at which some basic variable becomes negative, we apply the dual simplex method in order to recover primal feasibility.

5.6 Summary

In this chapter, we have studied the dependence of optimal solutions and of the optimal cost on the problem data, that is, on the entries of $\mathbf{A}$, $\mathbf{b}$, and $\mathbf{c}$. For many of the cases that we have examined, a common methodology was used. Subsequent to a change in the problem data, we first examine its effects on the feasibility and optimality conditions. If we wish the same basis to remain optimal, this leads us to certain limitations on the magnitude of the changes in the problem data. For larger changes, we no longer have an optimal basis and some remedial action (involving the primal or dual simplex method) is typically needed.

We close with a summary of our main results.

(a) If a new variable is added, we check its reduced cost and if it is negative, we add a new column to the tableau and proceed from there.

(b) If a new constraint is added, we check whether it is violated and if so, we form an auxiliary problem and its tableau, and proceed from there.

(c) If an entry of $\mathbf{b}$ or $\mathbf{c}$ is changed by δ, we obtain an interval of values of δ for which the same basis remains optimal.

(d) If an entry of $\mathbf{A}$ is changed by δ, a similar analysis is possible. However, this case is somewhat complicated if the change affects an entry of a basic column.

(e) Assuming that the dual problem is feasible, the optimal cost is a piecewise linear convex function of the vector $\mathbf{b}$ (for those $\mathbf{b}$ for which the primal is feasible). Furthermore, subgradients of the optimal cost function correspond to optimal solutions to the dual problem.

(f) Assuming that the primal problem is feasible, the optimal cost is a piecewise linear concave function of the vector $\mathbf{c}$ (for those $\mathbf{c}$ for which the primal has finite cost).

(g) If the cost vector is an affine function of a scalar parameter θ, there is a systematic procedure (parametric programming) for solving the problem for all values of θ. A similar procedure is possible if the vector $\mathbf{b}$ is an affine function of a scalar parameter.

5.7 Exercises

Exercise 5.1 Consider the same problem as in Example 5.1, for which we already have an optimal basis. Let us introduce the additional constraint $x_1 + x_2 = 3$. Form the auxiliary problem described in the text, and solve it using the primal simplex method. Whenever the "large" constant M is compared to another number, M should be treated as being the larger one.

Exercise 5.2 (Sensitivity with respect to changes in a basic column of A) In this problem (and the next two) we study the change in the value of the optimal cost when an entry of the matrix $\mathbf{A}$ is perturbed by a small amount. We consider a linear programming problem in standard form, under the usual assumption that $\mathbf{A}$ has linearly independent rows. Suppose that we have an optimal basis $\mathbf{B}$ that leads to a nondegenerate optimal solution $\mathbf{x}^*$, and a nondegenerate dual optimal solution $\mathbf{p}$. We assume that the first column is basic. We will now change the first entry of $\mathbf{A}_1$ from a_{11} to $a_{11} + \delta$, where δ is a small scalar. Let $\mathbf{E}$ be a matrix of dimensions $m \times m$ (where m is the number of rows of $\mathbf{A}$), whose entries are all zero except for the top left entry e_{11}, which is equal to 1.

(a) Show that if δ is small enough, $\mathbf{B}+\delta\mathbf{E}$ is a basis matrix for the new problem.

(b) Show that under the basis $\mathbf{B} + \delta\mathbf{E}$, the vector $\mathbf{x}_B$ of basic variables in the new problem is equal to $(\mathbf{I} + \delta\mathbf{B}^{-1}\mathbf{E})^{-1}\mathbf{B}^{-1}\mathbf{b}$.

(c) Show that if δ is sufficiently small, $\mathbf{B} + \delta\mathbf{E}$ is an optimal basis for the new problem.

(d) We use the symbol $\approx$ to denote equality when second order terms in δ are ignored. The following approximation is known to be true: $(\mathbf{I}+\delta\mathbf{B}^{-1}\mathbf{E})^{-1} \approx \mathbf{I} - \delta\mathbf{B}^{-1}\mathbf{E}$. Using this approximation, show that

$$\mathbf{c}'_B\mathbf{x}_B \approx \mathbf{c}'\mathbf{x}^* - \delta p_1 x_1^*,$$

where x_1^* (respectively, p_1) is the first component of the optimal solution to the original primal (respectively, dual) problem, and $\mathbf{x}_B$ has been defined in part (b).

Exercise 5.3 (Sensitivity with respect to changes in a basic column of A) Consider a linear programming problem in standard form under the usual assumption that the rows of the matrix $\mathbf{A}$ are linearly independent. Suppose that the columns $\mathbf{A}_1, \ldots, \mathbf{A}_m$ form an optimal basis. Let $\mathbf{A}_0$ be some vector and suppose that we change $\mathbf{A}_1$ to $\mathbf{A}_1 + \delta\mathbf{A}_0$. Consider the matrix $\mathbf{B}(\delta)$ consisting of the columns $\mathbf{A}_0 + \delta\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_m$. Let $[\delta_1, \delta_2]$ be a closed interval of values of δ that contains zero and in which the determinant of $\mathbf{B}(\delta)$ is nonzero. Show that the subset of $[\delta_1, \delta_2]$ for which $\mathbf{B}(\delta)$ is an optimal basis is also a closed interval.

Exercise 5.4 Consider the problem in Example 5.1, with a_{11} changed from 3 to $3 + \delta$. Let us keep x_1 and x_2 as the basic variables and let $\mathbf{B}(\delta)$ be the corresponding basis matrix, as a function of δ.

(a) Compute $\mathbf{B}(\delta)^{-1}\mathbf{b}$. For which values of δ is $\mathbf{B}(\delta)$ a feasible basis?

(b) Compute $\mathbf{c}'_B\mathbf{B}(\delta)^{-1}$. For which values of δ is $\mathbf{B}(\delta)$ an optimal basis?

(c) Determine the optimal cost, as a function of δ, when δ is restricted to those values for which $\mathbf{B}(\delta)$ is an optimal basis matrix.

Exercise 5.5 While solving a standard form linear programming problem using the simplex method, we arrive at the following tableau:

		x_1	x_2	x_3	x_4	x_5
		0	0	$\bar{c}_3$	0	$\bar{c}_5$
$x_2 =$	1	0	1	-1	0	β
$x_4 =$	2	0	0	2	1	γ
$x_1 =$	3	1	0	4	0	δ

Suppose also that the last three columns of the matrix $\mathbf{A}$ form an identity matrix.

(a) Give necessary and sufficient conditions for the basis described by this tableau to be optimal (in terms of the coefficients in the tableau).

(b) Assume that this basis is optimal and that $\bar{c}_3 = 0$. Find an optimal basic feasible solution, other than the one described by this tableau.

(c) Suppose that $\gamma \geq 0$. Show that there exists an optimal basic feasible solution, regardless of the values of $\bar{c}_3$ and $\bar{c}_5$.

(d) Assume that the basis associated with this tableau is optimal. Suppose also that b_1 in the original problem is replaced by $b_1 + \epsilon$. Give upper and lower bounds on ϵ so that this basis remains optimal.

(e) Assume that the basis associated with this tableau is optimal. Suppose also that c_1 in the original problem is replaced by $c_1 + \epsilon$. Give upper and lower bounds on ϵ so that this basis remains optimal.

Exercise 5.6 Company A has agreed to supply the following quantities of special lamps to Company B during the next 4 months:

Month	January	February	March	April
Units	150	160	225	180

Company A can produce a maximum of 160 lamps per month at a cost of \$35 per unit. Additional lamps can be purchased from Company C at a cost of \$50 per lamp. Company A incurs an inventory holding cost of \$5 per month for each lamp held in inventory.

(a) Formulate the problem that Company A is facing as a linear programming problem.

(b) Solve the problem using a linear programming package.

(c) Company A is considering some preventive maintenance during one of the first three months. If maintenance is scheduled for January, the company can manufacture only 151 units (instead of 160); similarly, the maximum possible production if maintenance is scheduled for February or March is 153 and 155 units, respectively. What maintenance schedule would you recommend and why?

(d) Company D has offered to supply up to 50 lamps (total) to Company A during either January, February or March. Company D charges \$45 per lamp. Should Company A buy lamps from Company D? If yes, when and how many lamps should Company A purchase, and what is the impact of this decision on the total cost?

(e) Company C has offered to lower the price of units supplied to Company A during February. What is the maximum decrease that would make this offer attractive to Company A?

(f) Because of anticipated increases in interest rates, the holding cost per lamp is expected to increase to \$8 per unit in February. How does this change affect the total cost and the optimal solution?

(g) Company B has just informed Company A that it requires only 90 units in January (instead of 150 requested previously). Calculate upper and lower bounds on the impact of this order on the optimal cost using information from the optimal solution to the original problem.

Exercise 5.7 A paper company manufactures three basic products: pads of paper, 5-packs of paper, and 20-packs of paper. The pad of paper consists of a single pad of 25 sheets of lined paper. The 5-pack consists of 5 pads of paper, together with a small notebook. The 20-pack of paper consists of 20 pads of paper, together with a large notebook. The small and large notebooks are not sold separately.

Production of each pad of paper requires 1 minute of paper-machine time, 1 minute of supervisory time, and \$.10 in direct costs. Production of each small notebook takes 2 minutes of paper-machine time, 45 seconds of supervisory time, and \$.20 in direct cost. Production of each large notebook takes 3 minutes of paper machine time, 30 seconds of supervisory time and \$.30 in direct costs. To package the 5-pack takes 1 minute of packager's time and 1 minute of supervisory time. To package the 20-pack takes 3 minutes of packager's time and 2 minutes of supervisory time. The amounts of available paper-machine time, supervisory time, and packager's time are constants b_1, b_2, b_3, respectively. Any of the three products can be sold to retailers in any quantity at the prices \$.30, \$1.60, and \$7.00, respectively.

Provide a linear programming formulation of the problem of determining an optimal mix of the three products. (You may ignore the constraint that only integer quantities can be produced.) Try to formulate the problem in such a way that the following questions can be answered by looking at a single dual variable or reduced cost in the final tableau. Also, for each question, give a brief explanation of why it can be answered by looking at just one dual price or reduced cost.

(a) What is the marginal value of an extra unit of supervisory time?

(b) What is the lowest price at which it is worthwhile to produce single pads of paper for sale?

(c) Suppose that part-time supervisors can be hired at \$8 per hour. Is it worthwhile to hire any?

(d) Suppose that the direct cost of producing pads of paper increases from \$.10 to \$.12. What is the profit decrease?

Exercise 5.8 A pottery manufacturer can make four different types of dining room service sets: JJP English, Currier, Primrose, and Bluetail. Furthermore, Primrose can be made by two different methods. Each set uses clay, enamel, dry room time, and kiln time, and results in a profit shown in Table 5.3. (Here, lbs is the abbreviation for pounds).

Resources	E	C	P_1	P_2	B	Total
Clay (lbs)	10	15	10	10	20	130
Enamel (lbs)	1	2	2	1	1	13
Dry room (hours)	3	1	6	6	3	45
Kiln (hours)	2	4	2	5	3	23
Profit	51	102	66	66	89	

Table 5.3: The rightmost column in the table gives the manufacturer's resource availability for the remainder of the week. Notice that Primrose can be made by two different methods. They both use the same amount of clay (10 lbs.) and dry room time (6 hours). But the second method uses one pound less of enamel and three more hours in the kiln.

The manufacturer is currently committed to making the same amount of Primrose using methods 1 and 2. The formulation of the profit maximization problem is given below. The decision variables E, C, P_1, P_2, B are the number of sets of type English, Currier, Primrose Method 1, Primrose Method 2, and Bluetail, respectively. We assume, for the purposes of this problem, that the number of sets of each type can be fractional.

$$
\begin{array}{llllllll}
\text{maximize} & 51E & + 102C & + 66P_1 & + 66P_2 & + 89B & & \\
\text{subject to} & 10E & + 15C & + 10P_1 & + 10P_2 & + 20B & \leq & 130 \\
 & E & + 2C & + 2P_1 & + P_2 & + B & \leq & 13 \\
 & 3E & + C & + 6P_1 & + 6P_2 & + 3B & \leq & 45 \\
 & 2E & + 4C & + 2P_1 & + 5P_2 & + 3B & \leq & 23 \\
 & & & P_1 & - P_2 & & = & 0 \\
 & \multicolumn{7}{l}{E, C, P_1, P_2, B \geq 0.}
\end{array}
$$

The optimal solution to the primal and the dual, respectively, together with sensitivity information, is given in Tables 5.4 and 5.5. Use this information to answer the questions that follow.

	Optimal Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
E	0	−3.571	51	3.571	∞
C	2	0	102	16.667	12.5
P_1	0	0	66	37.571	∞
P_2	0	−37.571	66	37.571	∞
B	5	0	89	47	12.5

Table 5.4: The optimal primal solution and its sensitivity with respect to changes in coefficients of the objective function. The last two columns describe the allowed changes in these coefficients for which the same solution remains optimal.

(a) What is the optimal quantity of each service set, and what is the total profit?

(b) Give an economic (not mathematical) interpretation of the optimal dual variables appearing in the sensitivity report, for each of the five constraints.

(c) Should the manufacturer buy an additional 20 lbs. of Clay at \$1.1 per pound?

(d) Suppose that the number of hours available in the dry room decreases by 30. Give a bound for the decrease in the total profit.

(e) In the current model, the number of Primrose produced using method 1 was required to be the same as the number of Primrose produced by method 2. Consider a revision of the model in which this constraint is replaced by the constraint $P_1 - P_2 \geq 0$. In the reformulated problem would the amount of Primrose made by method 1 be positive?

Exercise 5.9 Using the notation of Section 5.2, show that for any positive scalar λ and any $\mathbf{b} \in S$, we have $F(\lambda \mathbf{b}) = \lambda F(\mathbf{b})$. Assume that the dual feasible set is nonempty, so that $F(\mathbf{b})$ is finite.

	Slack Value	Dual Variable	Constr. RHS	Allowable Increase	Allowable Decrease
Clay	130	1.429	130	23.33	43.75
Enamel	9	0	13	∞	4
Dry Rm.	17	0	45	∞	28
Kiln	23	20.143	23	5.60	3.50
Prim.	0	11.429	0	3.50	0

Table 5.5: The optimal dual solution and its sensitivity. The column labeled "slack value" gives us the optimal values of the slack variables associated with each of the primal constraints. The third column simply repeats the right-hand side vector **b**, while the last two columns describe the allowed changes in the components of **b** for which the optimal dual solution remains the same.

Exercise 5.10 Consider the linear programming problem:

$$
\begin{array}{ll}
\text{minimize} & x_1 + x_2 \\
\text{subject to} & x_1 + 2x_2 = \theta, \\
 & x_1, x_2 \geq 0.
\end{array}
$$

(a) Find (by inspection) an optimal solution, as a function of θ.

(b) Draw a graph showing the optimal cost as a function of θ.

(c) Use the picture in part (b) to obtain the set of all dual optimal solutions, for every value of θ.

Exercise 5.11 Consider the function $g(\theta)$, as defined in the beginning of Section 5.5. Suppose that $g(\theta)$ is linear for $\theta \in [\theta_1, \theta_2]$. Is it true that there exists a unique optimal solution when $\theta_1 < \theta < \theta_2$? Prove or provide a counterexample.

Exercise 5.12 Consider the parametric programming problem discussed in Section 5.5.

(a) Suppose that for some value of θ, there are exactly two distinct basic feasible solutions that are optimal. Show that they must be adjacent.

(b) Let θ^* be a breakpoint of the function $g(\theta)$. Let $\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3$ be basic feasible solutions, all of which are optimal for $\theta = \theta^*$. Suppose that $\mathbf{x}^1$ is a unique optimal solution for $\theta < \theta^*$, $\mathbf{x}^3$ is a unique optimal solution for $\theta > \theta^*$, and $\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3$ are the only optimal basic feasible solutions for $\theta = \theta^*$. Provide an example to show that $\mathbf{x}^1$ and $\mathbf{x}^3$ need not be adjacent.

Exercise 5.13 Consider the following linear programming problem:

$$\begin{array}{lrcrcrcrcl} \text{minimize} & 4x_1 & & & + & 5x_3 & & & & \\ \text{subject to} & 2x_1 & + & x_2 & - & 5x_3 & & & = & 1 \\ & -3x_1 & & & + & 4x_3 & + & x_4 & = & 2 \\ & \multicolumn{9}{l}{x_1, x_2, x_3, x_4 \geq 0.} \end{array}$$

(a) Write down a simplex tableau and find an optimal solution. Is it unique?

(b) Write down the dual problem and find an optimal solution. Is it unique?

(c) Suppose now that we change the vector $\mathbf{b}$ from $\mathbf{b} = (1, 2)$ to $\mathbf{b} = (1 - 2\theta, 2 - 3\theta)$, where θ is a scalar parameter. Find an optimal solution and the value of the optimal cost, as a function of θ. (For all θ, both positive and negative.)

Exercise 5.14 Consider the problem

$$\begin{array}{ll} \text{minimize} & (\mathbf{c} + \theta\mathbf{d})'\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} = \mathbf{b} + \theta\mathbf{f} \\ & \mathbf{x} \geq \mathbf{0}, \end{array}$$

where $\mathbf{A}$ is an $m \times n$ matrix with linearly independent rows. We assume that the problem is feasible and the optimal cost $f(\theta)$ is finite for all values of θ in some interval $[\theta_1, \theta_2]$.

(a) Suppose that a certain basis is optimal for $\theta = -10$ and for $\theta = 10$. Prove that the same basis is optimal for $\theta = 5$.

(b) Show that $f(\theta)$ is a piecewise quadratic function of θ. Give an upper bound on the number of "pieces."

(c) Let $\mathbf{b} = \mathbf{0}$ and $\mathbf{c} = \mathbf{0}$. Suppose that a certain basis is optimal for $\theta = 1$. For what other nonnegative values of θ is that same basis optimal?

(d) Is $f(\theta)$ convex, concave or neither?

Exercise 5.15 Consider the problem

$$\begin{array}{ll} \text{minimize} & \mathbf{c}'\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} = \mathbf{b} + \theta\mathbf{d} \\ & \mathbf{x} \geq \mathbf{0}, \end{array}$$

and let $f(\theta)$ be the optimal cost, as a function of θ.

(a) Let $X(\theta)$ be the set of all optimal solutions, for a given value of θ. For any nonnegative scalar t, define $X(0, t)$ to be the union of the sets $X(\theta)$, $0 \leq \theta \leq t$. Is $X(0, t)$ a convex set? Provide a proof or a counterexample.

(b) Suppose that we remove the nonnegativity constraints $\mathbf{x} \geq \mathbf{0}$ from the problem under consideration. Is $X(0, t)$ a convex set? Provide a proof or a counterexample.

(c) Suppose that $\mathbf{x}^1$ and $\mathbf{x}^2$ belong to $X(0, t)$. Show that there is a continuous path from $\mathbf{x}^1$ to $\mathbf{x}^2$ that is contained within $X(0, t)$. That is, there exists a continuous function $g(\lambda)$ such that $g(\lambda_1) = \mathbf{x}^1$, $g(\lambda_2) = \mathbf{x}^2$, and $g(\lambda) \in X(0, t)$ for all $\lambda \in (\lambda_1, \lambda_2)$.

Exercise 5.16 Consider the parametric programming problem of Section 5.5. Suppose that some basic feasible solution is optimal if and only if θ is equal to some θ^*.

(a) Suppose that the feasible set is unbounded. Is it true that there exist at least three distinct basic feasible solutions that are optimal when $\theta = \theta^*$?

(b) Answer the question in part (a) for the case where the feasible set is bounded.

Exercise 5.17 Consider the parametric programming problem. Suppose that every basic solution encountered by the algorithm is nondegenerate. Prove that the algorithm does not cycle.

5.8 Notes and sources

The material in this chapter, with the exception of Section 5.3, is standard, and can be found in any text on linear programming.

5.1. A more detailed discussion of the results of the production planning case study can be found in Freund and Shannahan (1992).

5.3. The results in this section have beautiful generalizations to the case of nonlinear convex optimization; see, e.g., Rockafellar (1970).

5.5. Anticycling rules for parametric programming can be found in Murty (1983).