nally, Schrijver (1986) is a comprehensive, but more advanced reference on the subject.

1.1. The formulation of the diet problem is due to Stigler (1945).

1.2. The case study on DEC's production planning was developed by Freund and Shannahan (1992). Methods for dealing with the nurse scheduling and other cyclic problems are studied by Bartholdi, Orlin, and Ratliff (1980). More information on pattern classification can be found in Duda and Hart (1973), or Haykin (1994).

1.3. A deep and comprehensive treatment of convex functions and their properties is provided by Rockafellar (1970). Linear programming arises in control problems, in ways that are more sophisticated than what is described here; see, e.g., Dahleh and Diaz-Bobillo (1995).

1.5. For an introduction to linear algebra, see Strang (1988).

1.6. For a more detailed treatment of algorithms and their computational requirements, see Lewis and Papadimitriou (1981), Papadimitriou and Steiglitz (1982), or Cormen, Leiserson, and Rivest (1990).

1.7. Exercise 1.8 is adapted from Boyd and Vandenberghe (1995). Exercises 1.9 and 1.14 are adapted from Bradley, Hax, and Magnanti (1977). Exercise 1.11 is adapted from Ahuja, Magnanti, and Orlin (1993).

Chapter 2

The geometry of linear programming

Contents

In this chapter, we define a polyhedron as a set described by a finite number of linear equality and inequality constraints. In particular, the feasible set in a linear programming problem is a polyhedron. We study the basic geometric properties of polyhedra in some detail, with emphasis on their "corner points" (vertices). As it turns out, common geometric intuition derived from the familiar three-dimensional polyhedra is essentially correct when applied to higher-dimensional polyhedra. Another interesting aspect of the development in this chapter is that certain concepts (e.g., the concept of a vertex) can be defined either geometrically or algebraically. While the geometric view may be more natural, the algebraic approach is essential for carrying out computations. Much of the richness of the subject lies in the interplay between the geometric and the algebraic points of view.

Our development starts with a characterization of the corner points of feasible sets in the general form $\{\mathbf{x} \mid \mathbf{A}\mathbf{x} \geq \mathbf{b}\}$. Later on, we focus on the case where the feasible set is in the standard form $\{\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$, and we derive a simple algebraic characterization of the corner points. The latter characterization will play a central role in the development of the simplex method in Chapter 3.

The main results of this chapter state that a nonempty polyhedron has at least one corner point if and only if it does not contain a line, and if this is the case, the search for optimal solutions to linear programming problems can be restricted to corner points. These results are proved for the most general case of linear programming problems using geometric arguments. The same results will also be proved in the next chapter, for the case of standard form problems, as a corollary of our development of the simplex method. Thus, the reader who wishes to focus on standard form problems may skip the proofs in Sections 2.5 and 2.6. Finally, Sections 2.7 and 2.8 can also be skipped during a first reading; any results in these sections that are needed later on will be rederived in Chapter 4, using different techniques.

2.1 Polyhedra and convex sets

In this section, we introduce some important concepts that will be used to study the geometry of linear programming, including a discussion of convexity.

Hyperplanes, halfspaces, and polyhedra

We start with the formal definition of a polyhedron.

> **Definition 2.1** *A* **polyhedron** *is a set that can be described in the form* $\{\mathbf{x} \in \Re^n \mid \mathbf{A}\mathbf{x} \geq \mathbf{b}\}$, *where* $\mathbf{A}$ *is an* $m \times n$ *matrix and* $\mathbf{b}$ *is a vector in* $\Re^m$.

As discussed in Section 1.1, the feasible set of any linear programming problem can be described by inequality constraints of the form $\mathbf{A}\mathbf{x} \geq \mathbf{b}$, and is therefore a polyhedron. In particular, a set of the form $\{\mathbf{x} \in \Re^n \mid \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$ is also a polyhedron and will be referred to as a *polyhedron in standard form*.

A polyhedron can either "extend to infinity," or can be confined in a finite region. The definition that follows refers to this distinction.

> **Definition 2.2** *A set* $S \subset \Re^n$ *is* **bounded** *if there exists a constant* K *such that the absolute value of every component of every element of* S *is less than or equal to* K.

The next definition deals with polyhedra determined by a single linear constraint.

> **Definition 2.3** *Let* $\mathbf{a}$ *be a nonzero vector in* $\Re^n$ *and let* b *be a scalar.*
>
> **(a)** *The set* $\{\mathbf{x} \in \Re^n \mid \mathbf{a}'\mathbf{x} = b\}$ *is called a* **hyperplane**.
>
> **(b)** *The set* $\{\mathbf{x} \in \Re^n \mid \mathbf{a}'\mathbf{x} \geq b\}$ *is called a* **halfspace**.

Note that a hyperplane is the boundary of a corresponding halfspace. In addition, the vector $\mathbf{a}$ in the definition of the hyperplane is perpendicular to the hyperplane itself. [To see this, note that if $\mathbf{x}$ and $\mathbf{y}$ belong to the same hyperplane, then $\mathbf{a}'\mathbf{x} = \mathbf{a}'\mathbf{y}$. Hence, $\mathbf{a}'(\mathbf{x} - \mathbf{y}) = 0$ and therefore $\mathbf{a}$ is orthogonal to any direction vector confined to the hyperplane.] Finally, note that a polyhedron is equal to the intersection of a finite number of halfspaces; see Figure 2.1.

Convex Sets

We now define the important notion of a convex set.

> **Definition 2.4** *A set* $S \subset \Re^n$ *is* **convex** *if for any* $\mathbf{x}, \mathbf{y} \in S$, *and any* $\lambda \in [0, 1]$, *we have* $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in S$.

Note that if $\lambda \in [0, 1]$, then $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}$ is a weighted average of the vectors $\mathbf{x}$, $\mathbf{y}$, and therefore belongs to the line segment joining $\mathbf{x}$ and $\mathbf{y}$. Thus, a set is convex if the segment joining any two of its elements is contained in the set; see Figure 2.2.

Our next definition refers to weighted averages of a finite number of vectors; see Figure 2.3.

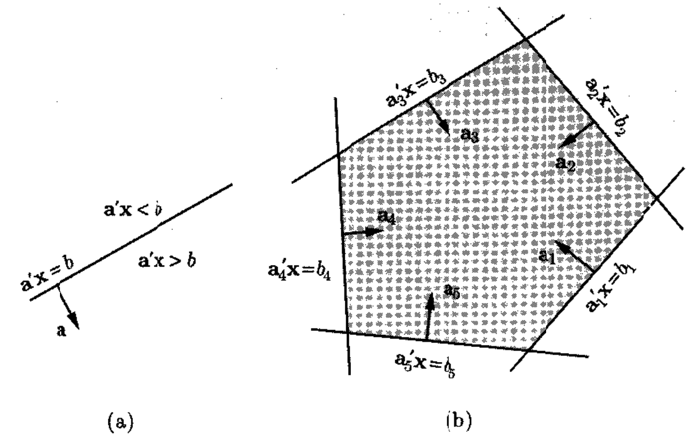

Figure 2.1: (a) A hyperplane and two halfspaces. (b) The polyhedron $\{\mathbf{x} \mid \mathbf{a}_i'\mathbf{x} \geq b_i,\ i = 1, \ldots, 5\}$ is the intersection of five halfspaces. Note that each vector $\mathbf{a}_i$ is perpendicular to the hyperplane $\{\mathbf{x} \mid \mathbf{a}_i'\mathbf{x} = b_i\}$.

Definition 2.5 *Let $\mathbf{x}^1, \ldots, \mathbf{x}^k$ be vectors in $\Re^n$ and let $\lambda_1, \ldots, \lambda_k$ be nonnegative scalars whose sum is unity.*

(a) *The vector $\sum_{i=1}^{k} \lambda_i \mathbf{x}^i$ is said to be a* **convex combination** *of the vectors $\mathbf{x}^1, \ldots, \mathbf{x}^k$.*

(b) *The* **convex hull** *of the vectors $\mathbf{x}^1, \ldots, \mathbf{x}^k$ is the set of all convex combinations of these vectors.*

The result that follows establishes some important facts related to convexity.

Theorem 2.1

(a) *The intersection of convex sets is convex.*

(b) *Every polyhedron is a convex set.*

(c) *A convex combination of a finite number of elements of a convex set also belongs to that set.*

(d) *The convex hull of a finite number of vectors is a convex set.*

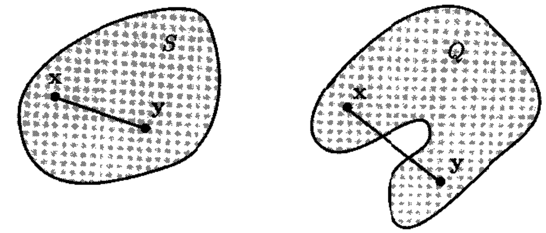

Figure 2.2: The set S is convex, but the set Q is not, because the segment joining $\mathbf{x}$ and $\mathbf{y}$ is not contained in Q.

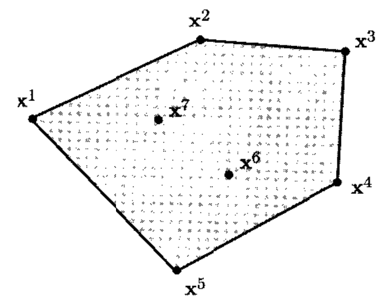

Figure 2.3: The convex hull of seven points in $\Re^2$.

Proof.

(a) Let S_i, $i \in I$, be convex sets where I is some index set, and suppose that $\mathbf{x}$ and $\mathbf{y}$ belong to the intersection $\cap_{i \in I} S_i$. Let $\lambda \in [0, 1]$. Since each S_i is convex and contains $\mathbf{x}, \mathbf{y}$, we have $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in S_i$, which proves that $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ also belongs to the intersection of the sets S_i. Therefore, $\cap_{i \in I} S_i$ is convex.

(b) Let $\mathbf{a}$ be a vector and let b a scalar. Suppose that $\mathbf{x}$ and $\mathbf{y}$ satisfy $\mathbf{a}'\mathbf{x} \geq b$ and $\mathbf{a}'\mathbf{y} \geq b$, respectively, and therefore belong to the same halfspace. Let $\lambda \in [0, 1]$. Then, $\mathbf{a}'(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \geq \lambda b + (1-\lambda)b = b$, which proves that $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ also belongs to the same halfspace. Therefore a halfspace is convex. Since a polyhedron is the intersection of a finite number of halfspaces, the result follows from part (a).

(c) A convex combination of two elements of a convex set lies in that set, by the definition of convexity. Let us assume, as an induction hypothesis, that a convex combination of k elements of a convex set belongs to that set. Consider $k+1$ elements $\mathbf{x}^1, \ldots, \mathbf{x}^{k+1}$ of a convex set S and let $\lambda_1, \ldots, \lambda_{k+1}$ be nonnegative scalars that sum to 1. We assume, without loss of generality, that $\lambda_{k+1} \neq 1$. We then have

$$\sum_{i=1}^{k+1} \lambda_i \mathbf{x}^i = \lambda_{k+1}\mathbf{x}^{k+1} + (1-\lambda_{k+1}) \sum_{i=1}^{k} \frac{\lambda_i}{1-\lambda_{k+1}} \mathbf{x}^i. \tag{2.1}$$

The coefficients $\lambda_i/(1-\lambda_{k+1})$, $i = 1, \ldots, k$, are nonnegative and sum to unity; using the induction hypothesis, $\sum_{i=1}^{k} \lambda_i \mathbf{x}^i/(1-\lambda_{k+1}) \in S$. Then, the fact that S is convex and Eq. (2.1) imply that $\sum_{i=1}^{k+1} \lambda_i \mathbf{x}^i \in S$, and the induction step is complete.

(d) Let S be the convex hull of the vectors $\mathbf{x}^1, \ldots, \mathbf{x}^k$ and let $\mathbf{y} = \sum_{i=1}^{k} \zeta_i \mathbf{x}^i$, $\mathbf{z} = \sum_{i=1}^{k} \theta_i \mathbf{x}^i$ be two elements of S, where $\zeta_i \geq 0$, $\theta_i \geq 0$, and $\sum_{i=1}^{k} \zeta_i = \sum_{i=1}^{k} \theta_i = 1$. Let $\lambda \in [0,1]$. Then,

$$\lambda \mathbf{y} + (1-\lambda)\mathbf{z} = \lambda \sum_{i=1}^{k} \zeta_i \mathbf{x}^i + (1-\lambda) \sum_{i=1}^{k} \theta_i \mathbf{x}^i = \sum_{i=1}^{k} \big(\lambda\zeta_i + (1-\lambda)\theta_i\big)\mathbf{x}^i.$$

We note that the coefficients $\lambda\zeta_i + (1-\lambda)\theta_i$, $i = 1, \ldots, k$, are nonnegative and sum to unity. This shows that $\lambda\mathbf{y} + (1-\lambda)\mathbf{z}$ is a convex combination of $\mathbf{x}^1, \ldots, \mathbf{x}^k$ and, therefore, belongs to S. This establishes the convexity of S. □

2.2 Extreme points, vertices, and basic feasible solutions

We observed in Section 1.4 that an optimal solution to a linear programming problem tends to occur at a "corner" of the polyhedron over which we are optimizing. In this section, we suggest three different ways of defining the concept of a "corner" and then show that all three definitions are equivalent.

Our first definition defines an *extreme point* of a polyhedron as a point that cannot be expressed as a convex combination of two other elements of the polyhedron, and is illustrated in Figure 2.4. Notice that this definition is entirely geometric and does not refer to a specific representation of a polyhedron in terms of linear constraints.

Definition 2.6 *Let P be a polyhedron. A vector $\mathbf{x} \in P$ is an* **extreme point** *of P if we cannot find two vectors $\mathbf{y}, \mathbf{z} \in P$, both different from $\mathbf{x}$, and a scalar $\lambda \in [0,1]$, such that $\mathbf{x} = \lambda\mathbf{y} + (1-\lambda)\mathbf{z}$.*

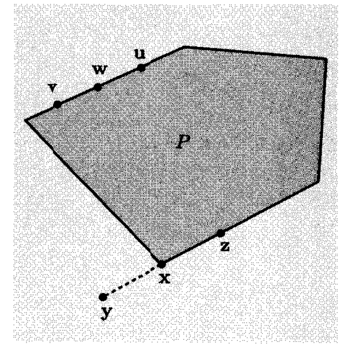

Figure 2.4: The vector $\mathbf{w}$ is not an extreme point because it is a convex combination of $\mathbf{v}$ and $\mathbf{u}$. The vector $\mathbf{x}$ is an extreme point: if $\mathbf{x} = \lambda\mathbf{y} + (1-\lambda)\mathbf{z}$ and $\lambda \in [0,1]$, then either $\mathbf{y} \notin P$, or $\mathbf{z} \notin P$, or $\mathbf{x} = \mathbf{y}$, or $\mathbf{x} = \mathbf{z}$.

An alternative geometric definition defines a *vertex* of a polyhedron P as the unique optimal solution to some linear programming problem with feasible set P.

Definition 2.7 *Let P be a polyhedron. A vector $\mathbf{x} \in P$ is a* **vertex** *of P if there exists some $\mathbf{c}$ such that $\mathbf{c}'\mathbf{x} < \mathbf{c}'\mathbf{y}$ for all $\mathbf{y}$ satisfying $\mathbf{y} \in P$ and $\mathbf{y} \neq \mathbf{x}$.*

In other words, $\mathbf{x}$ is a vertex of P if and only if P is on one side of a hyperplane (the hyperplane $\{\mathbf{y} \mid \mathbf{c}'\mathbf{y} = \mathbf{c}'\mathbf{x}\}$) which meets P only at the point $\mathbf{x}$; see Figure 2.5.

The two geometric definitions that we have given so far are not easy to work with from an algorithmic point of view. We would like to have a definition that relies on a representation of a polyhedron in terms of linear constraints and which reduces to an algebraic test. In order to provide such a definition, we need some more terminology.

Consider a polyhedron $P \subset \Re^n$ defined in terms of the linear equality and inequality constraints

$$\begin{aligned} \mathbf{a}_i'\mathbf{x} &\geq b_i, & i &\in M_1, \\ \mathbf{a}_i'\mathbf{x} &\leq b_i, & i &\in M_2, \\ \mathbf{a}_i'\mathbf{x} &= b_i, & i &\in M_3, \end{aligned}$$

where M_1, M_2, and M_3 are finite index sets, each $\mathbf{a}_i$ is a vector in $\Re^n$, and

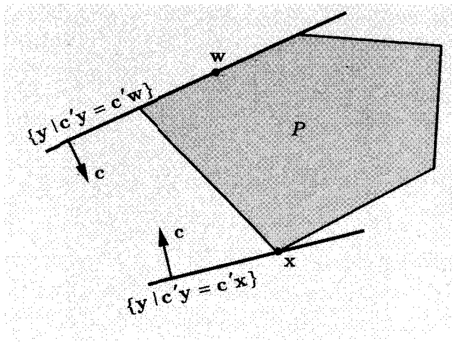

Figure 2.5: The line at the bottom touches P at a single point and $\mathbf{x}$ is a vertex. On the other hand, $\mathbf{w}$ is not a vertex because there is no hyperplane that meets P only at $\mathbf{w}$.

each b_i is a scalar. The definition that follows is illustrated in Figure 2.6.

> **Definition 2.8** *If a vector* $\mathbf{x}^*$ *satisfies* $\mathbf{a}_i'\mathbf{x}^* = b_i$ *for some* i *in* M_1, M_2, *or* M_3, *we say that the corresponding constraint is* **active** *or* **binding** *at* $\mathbf{x}^*$.

If there are n constraints that are active at a vector $\mathbf{x}^*$, then $\mathbf{x}^*$ satisfies a certain system of n linear equations in n unknowns. This system has a unique solution if and only if these n equations are "linearly independent." The result that follows gives a precise meaning to this statement, together with a slight generalization.

> **Theorem 2.2** *Let* $\mathbf{x}^*$ *be an element of* $\Re^n$ *and let* $I = \{i \mid \mathbf{a}_i'\mathbf{x}^* = b_i\}$ *be the set of indices of constraints that are active at* $\mathbf{x}^*$. *Then, the following are equivalent:*
>
> **(a)** *There exist* n *vectors in the set* $\{\mathbf{a}_i \mid i \in I\}$, *which are linearly independent.*
>
> **(b)** *The span of the vectors* $\mathbf{a}_i$, $i \in I$, *is all of* $\Re^n$, *that is, every element of* $\Re^n$ *can be expressed as a linear combination of the vectors* $\mathbf{a}_i$, $i \in I$.
>
> **(c)** *The system of equations* $\mathbf{a}_i'\mathbf{x} = b_i$, $i \in I$, *has a unique solution.*

Proof. Suppose that the vectors $\mathbf{a}_i$, $i \in I$, span $\Re^n$. Then, the span of these vectors has dimension n. By Theorem 1.3(a) in Section 1.5, n of

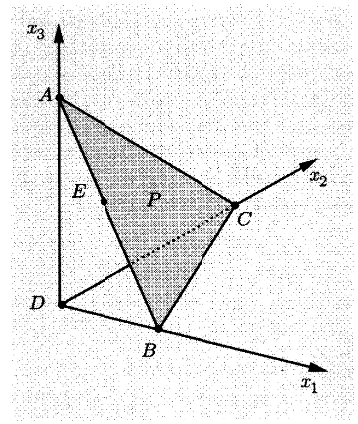

Figure 2.6: Let $P = \{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 1,\ x_1, x_2, x_3 \geq 0\}$. There are three constraints that are active at each one of the points A, B, C and D. There are only two constraints that are active at point E, namely $x_1 + x_2 + x_3 = 1$ and $x_2 = 0$.

these vectors form a basis of $\Re^n$, and are therefore linearly independent. Conversely, suppose that n of the vectors $\mathbf{a}_i$, $i \in I$, are linearly independent. Then, the subspace spanned by these n vectors is n-dimensional and must be equal to $\Re^n$. Hence, every element of $\Re^n$ is a linear combination of the vectors $\mathbf{a}_i$, $i \in I$. This establishes the equivalence of (a) and (b).

If the system of equations $\mathbf{a}_i'\mathbf{x} = b_i$, $i \in I$, has multiple solutions, say $\mathbf{x}^1$ and $\mathbf{x}^2$, then the nonzero vector $\mathbf{d} = \mathbf{x}^1 - \mathbf{x}^2$ satisfies $\mathbf{a}_i'\mathbf{d} = 0$ for all $i \in I$. Since $\mathbf{d}$ is orthogonal to every vector $\mathbf{a}_i$, $i \in I$, $\mathbf{d}$ is not a linear combination of these vectors and it follows that the vectors $\mathbf{a}_i$, $i \in I$, do not span $\Re^n$. Conversely, if the vectors $\mathbf{a}_i$, $i \in I$, do not span $\Re^n$, choose a nonzero vector $\mathbf{d}$ which is orthogonal to the subspace spanned by these vectors. If $\mathbf{x}$ satisfies $\mathbf{a}_i'\mathbf{x} = b_i$ for all $i \in I$, we also have $\mathbf{a}_i'(\mathbf{x} + \mathbf{d}) = b_i$ for all $i \in I$, thus obtaining multiple solutions. We have therefore established that (b) and (c) are equivalent. □

With a slight abuse of language, we will often say that certain *constraints* are *linearly independent*, meaning that the corresponding vectors $\mathbf{a}_i$ are linearly independent. With this terminology, statement (a) in Theorem 2.2 requires that there exist n linearly independent constraints that are active at $\mathbf{x}^*$.

We are now ready to provide an algebraic definition of a corner point, as a feasible solution at which there are n linearly independent active constraints. Note that since we are interested in a feasible solution, all equality constraints must be active. This suggests the following way of looking for corner points: first impose the equality constraints and then require that enough additional constraints be active, so that we get a total of n linearly independent active constraints. Once we have n linearly independent active constraints, a unique vector $\mathbf{x}^*$ is determined (Theorem 2.2). However, this procedure has no guarantee of leading to a feasible vector $\mathbf{x}^*$, because some of the inactive constraints could be violated; in the latter case we say that we have a basic (but not basic feasible) solution.

Definition 2.9 *Consider a polyhedron P defined by linear equality and inequality constraints, and let $\mathbf{x}^*$ be an element of $\Re^n$.*

(a) *The vector $\mathbf{x}^*$ is a* **basic solution** *if:*

- *(i) All equality constraints are active;*
- *(ii) Out of the constraints that are active at $\mathbf{x}^*$, there are n of them that are linearly independent.*

(b) *If $\mathbf{x}^*$ is a basic solution that satisfies all of the constraints, we say that it is a* **basic feasible solution**.

In reference to Figure 2.6, we note that points A, B, and C are basic feasible solutions. Point D is not a basic solution because it fails to satisfy the equality constraint. Point E is feasible, but not basic. If the equality constraint $x_1 + x_2 + x_3 = 1$ were to be replaced by the constraints $x_1 + x_2 + x_3 \leq 1$ and $x_1 + x_2 + x_3 \geq 1$, then D would be a basic solution, according to our definition. This shows that whether a point is a basic solution or not may depend on the way that a polyhedron is represented. Definition 2.9 is also illustrated in Figure 2.7.

Note that if the number m of constraints used to define a polyhedron $P \subset \Re^n$ is less than n, the number of active constraints at any given point must also be less than n, and there are no basic or basic feasible solutions.

We have given so far three different definitions that are meant to capture the same concept; two of them are geometric (extreme point, vertex) and the third is algebraic (basic feasible solution). Fortunately, all three definitions are equivalent as we prove next and, for this reason, the three terms can be used interchangeably.

Theorem 2.3 *Let P be a nonempty polyhedron and let $\mathbf{x}^* \in P$. Then, the following are equivalent:*

(a) *$\mathbf{x}^*$ is a vertex;*

(b) *$\mathbf{x}^*$ is an extreme point;*

(c) *$\mathbf{x}^*$ is a basic feasible solution.*

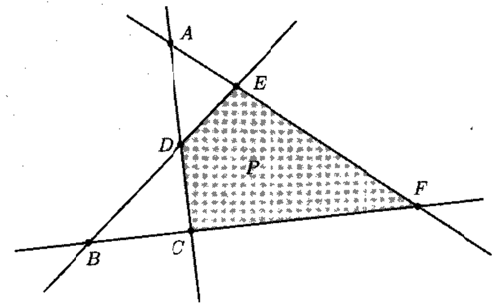

Figure 2.7: The points A, B, C, D, E, F are all basic solutions because at each one of them, there are two linearly independent constraints that are active. Points C, D, E, F are basic feasible solutions.

Proof. For the purposes of this proof and without loss of generality, we assume that P is represented in terms of constraints of the form $\mathbf{a}_i'\mathbf{x} \geq b_i$ and $\mathbf{a}_i'\mathbf{x} = b_i$.

Vertex $\Rightarrow$ Extreme point

Suppose that $\mathbf{x}^* \in P$ is a vertex. Then, by Definition 2.7, there exists some $\mathbf{c} \in \Re^n$ such that $\mathbf{c}'\mathbf{x}^* < \mathbf{c}'\mathbf{y}$ for every $\mathbf{y}$ satisfying $\mathbf{y} \in P$ and $\mathbf{y} \neq \mathbf{x}^*$. If $\mathbf{y} \in P$, $\mathbf{z} \in P$, $\mathbf{y} \neq \mathbf{x}^*$, $\mathbf{z} \neq \mathbf{x}^*$, and $0 \leq \lambda \leq 1$, then $\mathbf{c}'\mathbf{x}^* < \mathbf{c}'\mathbf{y}$ and $\mathbf{c}'\mathbf{x}^* < \mathbf{c}'\mathbf{z}$, which implies that $\mathbf{c}'\mathbf{x}^* < \mathbf{c}'(\lambda\mathbf{y} + (1-\lambda)\mathbf{z})$ and, therefore, $\mathbf{x}^* \neq \lambda\mathbf{y} + (1-\lambda)\mathbf{z}$. Thus, $\mathbf{x}^*$ cannot be expressed as a convex combination of two other elements of P and is, therefore, an extreme point (cf. Definition 2.6).

Extreme point $\Rightarrow$ Basic feasible solution

Suppose that $\mathbf{x}^* \in P$ is not a basic feasible solution. We will show that $\mathbf{x}^*$ is not an extreme point of P. Let $I = \{i \mid \mathbf{a}_i'\mathbf{x}^* = b_i\}$. Since $\mathbf{x}^*$ is not a basic feasible solution, there do not exist n linearly independent vectors in the family $\mathbf{a}_i$, $i \in I$. Thus, the vectors $\mathbf{a}_i$, $i \in I$, lie in a proper subspace of $\Re^n$, and there exists some nonzero vector $\mathbf{d} \in \Re^n$ such that $\mathbf{a}_i'\mathbf{d} = 0$, for all $i \in I$. Let ϵ be a small positive number and consider the vectors $\mathbf{y} = \mathbf{x}^* + \epsilon\mathbf{d}$ and $\mathbf{z} = \mathbf{x}^* - \epsilon\mathbf{d}$. Notice that $\mathbf{a}_i'\mathbf{y} = \mathbf{a}_i'\mathbf{x}^* = b_i$, for $i \in I$. Furthermore, for $i \notin I$, we have $\mathbf{a}_i'\mathbf{x}^* > b_i$ and, provided that ϵ is small, we will also have $\mathbf{a}_i'\mathbf{y} > b_i$. (It suffices to choose ϵ so that $\epsilon|\mathbf{a}_i'\mathbf{d}| < \mathbf{a}_i'\mathbf{x}^* - b_i$ for all $i \notin I$.) Thus, when ϵ is small enough, $\mathbf{y} \in P$ and, by a similar argument, $\mathbf{z} \in P$. We finally notice that $\mathbf{x}^* = (\mathbf{y} + \mathbf{z})/2$, which implies that $\mathbf{x}^*$ is not an extreme point.

Basic feasible solution ⇒ Vertex

Let $\mathbf{x}^*$ be a basic feasible solution and let $I = \{i \mid \mathbf{a}_i'\mathbf{x}^* = b_i\}$. Let $\mathbf{c} = \sum_{i \in I} \mathbf{a}_i$. We then have

$$\mathbf{c}'\mathbf{x}^* = \sum_{i \in I} \mathbf{a}_i'\mathbf{x}^* = \sum_{i \in I} b_i.$$

Furthermore, for any $\mathbf{x} \in P$ and any i, we have $\mathbf{a}_i'\mathbf{x} \geq b_i$, and

$$\mathbf{c}'\mathbf{x} = \sum_{i \in I} \mathbf{a}_i'\mathbf{x} \geq \sum_{i \in I} b_i. \tag{2.2}$$

This shows that $\mathbf{x}^*$ is an optimal solution to the problem of minimizing $\mathbf{c}'\mathbf{x}$ over the set P. Furthermore, equality holds in (2.2) if and only if $\mathbf{a}_i'\mathbf{x} = b_i$ for all $i \in I$. Since $\mathbf{x}^*$ is a basic feasible solution, there are n linearly independent constraints that are active at $\mathbf{x}^*$, and $\mathbf{x}^*$ is the unique solution to the system of equations $\mathbf{a}_i'\mathbf{x} = b_i$, $i \in I$ (Theorem 2.2). It follows that $\mathbf{x}^*$ is the unique minimizer of $\mathbf{c}'\mathbf{x}$ over the set P and, therefore, $\mathbf{x}^*$ is a vertex of P. □

Since a vector is a basic feasible solution if and only if it is an extreme point, and since the definition of an extreme point does not refer to any particular representation of a polyhedron, we conclude that the property of being a basic feasible solution is also independent of the representation used. (This is in contrast to the definition of a basic solution, which is representation dependent, as pointed out in the discussion that followed Definition 2.9.)

We finally note the following important fact.

Corollary 2.1 *Given a finite number of linear inequality constraints, there can only be a finite number of basic or basic feasible solutions.*

Proof. Consider a system of m linear inequality constraints imposed on a vector $\mathbf{x} \in \Re^n$. At any basic solution, there are n linearly independent active constraints. Since any n linearly independent active constraints define a unique point, it follows that different basic solutions correspond to different sets of n linearly independent active constraints. Therefore, the number of basic solutions is bounded above by the number of ways that we can choose n constraints out of a total of m, which is finite. □

Although the number of basic and, therefore, basic feasible solutions is guaranteed to be finite, it can be very large. For example, the unit cube $\{\mathbf{x} \in \Re^n \mid 0 \leq x_i \leq 1,\ i = 1, \ldots, n\}$ is defined in terms of $2n$ constraints, but has 2^n basic feasible solutions.

Adjacent basic solutions

Two distinct basic solutions to a set of linear constraints in $\Re^n$ are said to be *adjacent* if we can find $n-1$ linearly independent constraints that are active at both of them. In reference to Figure 2.7, D and E are adjacent to B; also, A and C are adjacent to D. If two adjacent basic solutions are also feasible, then the line segment that joins them is called an *edge* of the feasible set (see also Exercise 2.15).

2.3 Polyhedra in standard form

The definition of a basic solution (Definition 2.9) refers to general polyhedra. We will now specialize to polyhedra in standard form. The definitions and the results in this section are central to the development of the simplex method in the next chapter.

Let $P = \{\mathbf{x} \in \Re^n \mid \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$ be a polyhedron in standard form, and let the dimensions of $\mathbf{A}$ be $m \times n$, where m is the number of equality constraints. In most of our discussion of standard form problems, we will make the assumption that the m rows of the matrix $\mathbf{A}$ are linearly independent. (Since the rows are n-dimensional, this requires that $m \leq n$.) At the end of this section, we show that when P is nonempty, linearly dependent rows of $\mathbf{A}$ correspond to redundant constraints that can be discarded; therefore, our linear independence assumption can be made without loss of generality.

Recall that at any basic solution, there must be n linearly independent constraints that are active. Furthermore, every basic solution must satisfy the equality constraints $\mathbf{A}\mathbf{x} = \mathbf{b}$, which provides us with m active constraints; these are linearly independent because of our assumption on the rows of $\mathbf{A}$. In order to obtain a total of n active constraints, we need to choose $n - m$ of the variables x_i and set them to zero, which makes the corresponding nonnegativity constraints $x_i \geq 0$ active. However, for the resulting set of n active constraints to be linearly independent, the choice of these $n - m$ variables is not entirely arbitrary, as shown by the following result.

Theorem 2.4 *Consider the constraints $\mathbf{A}\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$ and assume that the $m \times n$ matrix $\mathbf{A}$ has linearly independent rows. A vector $\mathbf{x} \in \Re^n$ is a basic solution if and only if we have $\mathbf{A}\mathbf{x} = \mathbf{b}$, and there exist indices $B(1), \ldots, B(m)$ such that:*

(a) *The columns $\mathbf{A}_{B(1)}, \ldots, \mathbf{A}_{B(m)}$ are linearly independent;*

(b) *If $i \neq B(1), \ldots, B(m)$, then $x_i = 0$.*

Proof. Consider some $\mathbf{x} \in \Re^n$ and suppose that there are indices $B(1), \ldots,$

$B(m)$ that satisfy (a) and (b) in the statement of the theorem. The active constraints $x_i = 0$, $i \neq B(1), \ldots, B(m)$, and $\mathbf{Ax} = \mathbf{b}$ imply that

$$\sum_{i=1}^{m} \mathbf{A}_{B(i)} x_{B(i)} = \sum_{i=1}^{n} \mathbf{A}_i x_i = \mathbf{Ax} = \mathbf{b}.$$

Since the columns $\mathbf{A}_{B(i)}$, $i = 1, \ldots, m$, are linearly independent, $x_{B(1)}, \ldots, x_{B(m)}$ are uniquely determined. Thus, the system of equations formed by the active constraints has a unique solution. By Theorem 2.2, there are n linearly independent active constraints, and this implies that $\mathbf{x}$ is a basic solution.

For the converse, we assume that $\mathbf{x}$ is a basic solution and we will show that conditions (a) and (b) in the statement of the theorem are satisfied. Let $x_{B(1)}, \ldots, x_{B(k)}$ be the components of $\mathbf{x}$ that are nonzero. Since $\mathbf{x}$ is a basic solution, the system of equations formed by the active constraints $\sum_{i=1}^{n} \mathbf{A}_i x_i = \mathbf{b}$ and $x_i = 0$, $i \neq B(1), \ldots, B(k)$, have a unique solution (cf. Theorem 2.2); equivalently, the equation $\sum_{i=1}^{k} \mathbf{A}_{B(i)} x_{B(i)} = \mathbf{b}$ has a unique solution. It follows that the columns $\mathbf{A}_{B(1)}, \ldots, \mathbf{A}_{B(k)}$ are linearly independent. [If they were not, we could find scalars $\lambda_1, \ldots, \lambda_k$, not all of them zero, such that $\sum_{i=1}^{k} \mathbf{A}_{B(i)} \lambda_i = 0$. This would imply that $\sum_{i=1}^{k} \mathbf{A}_{B(i)} (x_{B(i)} + \lambda_i) = \mathbf{b}$, contradicting the uniqueness of the solution.]

We have shown that the columns $\mathbf{A}_{B(1)}, \ldots, \mathbf{A}_{B(k)}$ are linearly independent and this implies that $k \leq m$. Since $\mathbf{A}$ has m linearly independent rows, it also has m linearly independent columns, which span $\Re^m$. It follows [cf. Theorem 1.3(b) in Section 1.5] that we can find $m-k$ additional columns $\mathbf{A}_{B(k+1)}, \ldots, \mathbf{A}_{B(m)}$ so that the columns $\mathbf{A}_{B(i)}$, $i = 1, \ldots, m$, are linearly independent. In addition, if $i \neq B(1), \ldots, B(m)$, then $i \neq B(1), \ldots, B(k)$ (because $k \leq m$), and $x_i = 0$. Therefore, both conditions (a) and (b) in the statement of the theorem are satisfied. □

In view of Theorem 2.4, all basic solutions to a standard form polyhedron can be constructed according to the following procedure.

Procedure for constructing basic solutions

1. Choose m linearly independent columns $\mathbf{A}_{B(1)}, \ldots, \mathbf{A}_{B(m)}$.
2. Let $x_i = 0$ for all $i \neq B(1), \ldots, B(m)$.
3. Solve the system of m equations $\mathbf{Ax} = \mathbf{b}$ for the unknowns $x_{B(1)}, \ldots, x_{B(m)}$.

If a basic solution constructed according to this procedure is nonnegative, then it is feasible, and it is a basic feasible solution. Conversely, since every basic feasible solution is a basic solution, it can be obtained from this procedure. If $\mathbf{x}$ is a basic solution, the variables $x_{B(1)}, \ldots, x_{B(m)}$ are called

basic variables; the remaining variables are called *nonbasic*. The columns $\mathbf{A}_{B(1)}, \ldots, \mathbf{A}_{B(m)}$ are called the *basic columns* and, since they are linearly independent, they form a *basis* of $\Re^m$. We will sometimes talk about two bases being *distinct* or *different*; our convention is that distinct bases involve different sets $\{B(1), \ldots, B(m)\}$ of *basic indices*; if two bases involve the same set of indices in a different order, they will be viewed as one and the same basis.

By arranging the m basic columns next to each other, we obtain an $m \times m$ matrix $\mathbf{B}$, called a *basis matrix*. (Note that this matrix is invertible because the basic columns are required to be linearly independent.) We can similarly define a vector $\mathbf{x}_B$ with the values of the basic variables. Thus,

$$\mathbf{B} = \begin{bmatrix} | & | & & | \\ \mathbf{A}_{B(1)} & \mathbf{A}_{B(2)} & \cdots & \mathbf{A}_{B(m)} \\ | & | & & | \end{bmatrix}, \qquad \mathbf{x}_B = \begin{bmatrix} x_{B(1)} \\ \vdots \\ x_{B(m)} \end{bmatrix}.$$

The basic variables are determined by solving the equation $\mathbf{B}\mathbf{x}_B = \mathbf{b}$ whose unique solution is given by

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}.$$

Example 2.1 Let the constraint $\mathbf{Ax} = \mathbf{b}$ be of the form

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 6 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 8 \\ 12 \\ 4 \\ 6 \end{bmatrix}.$$

Let us choose $\mathbf{A}_4, \mathbf{A}_5, \mathbf{A}_6, \mathbf{A}_7$ as our basic columns. Note that they are linearly independent and the corresponding basis matrix is the identity. We then obtain the basic solution $\mathbf{x} = (0,0,0,8,12,4,6)$ which is nonnegative and, therefore, is a basic feasible solution. Another basis is obtained by choosing the columns $\mathbf{A}_3, \mathbf{A}_5, \mathbf{A}_6, \mathbf{A}_7$ (note that they are linearly independent). The corresponding basic solution is $\mathbf{x} = (0,0,4,0,-12,4,6)$, which is not feasible because $x_5 = -12 < 0$.

Suppose now that there was an eighth column $\mathbf{A}_8$, identical to $\mathbf{A}_7$. Then, the two sets of columns $\{\mathbf{A}_3, \mathbf{A}_5, \mathbf{A}_6, \mathbf{A}_7\}$ and $\{\mathbf{A}_3, \mathbf{A}_5, \mathbf{A}_6, \mathbf{A}_8\}$ coincide. On the other hand the corresponding sets of basic indices, which are $\{3,5,6,7\}$ and $\{3,5,6,8\}$, are different and we have two different bases, according to our conventions.

For an intuitive view of basic solutions, recall our interpretation of the constraint $\mathbf{Ax} = \mathbf{b}$, or $\sum_{i=1}^{n} \mathbf{A}_i x_i = \mathbf{b}$, as a requirement to synthesize the vector $\mathbf{b} \in \Re^m$ using the resource vectors $\mathbf{A}_i$ (Section 1.1). In a basic solution, we use only m of the resource vectors, those associated with the basic variables. Furthermore, in a basic feasible solution, this is accomplished using a nonnegative amount of each basic vector; see Figure 2.8.

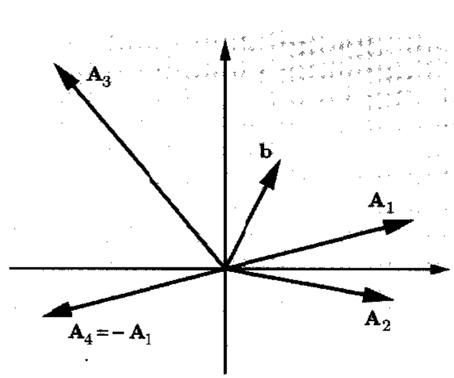

Figure 2.8: Consider a standard form problem with $n = 4$ and $m = 2$, and let the vectors $\mathbf{b}, \mathbf{A}_1, \ldots, \mathbf{A}_4$ be as shown. The vectors $\mathbf{A}_1, \mathbf{A}_2$ form a basis; the corresponding basic solution is infeasible because a negative value of x_2 is needed to synthesize $\mathbf{b}$ from $\mathbf{A}_1$, $\mathbf{A}_2$. The vectors $\mathbf{A}_1, \mathbf{A}_3$ form another basis; the corresponding basic solution is feasible. Finally, the vectors $\mathbf{A}_1, \mathbf{A}_4$ do not form a basis because they are linearly dependent.

Correspondence of bases and basic solutions

We now elaborate on the correspondence between basic solutions and bases. Different basic solutions must correspond to different bases, because a basis uniquely determines a basic solution. However, two different bases may lead to the same basic solution. (For an extreme example, if we have $\mathbf{b} = \mathbf{0}$, then every basis matrix leads to the same basic solution, namely, the zero vector.) This phenomenon has some important algorithmic implications, and is closely related to degeneracy, which is the subject of the next section.

Adjacent basic solutions and adjacent bases

Recall that two distinct basic solutions are said to be adjacent if there are $n - 1$ linearly independent constraints that are active at both of them. For standard form problems, we also say that two bases are *adjacent* if they share all but one basic column. Then, it is not hard to check that adjacent basic solutions can always be obtained from two adjacent bases. Conversely, if two adjacent bases lead to distinct basic solutions, then the latter are adjacent.

Example 2.2 In reference to Example 2.1, the bases $\{\mathbf{A}_4, \mathbf{A}_5, \mathbf{A}_6, \mathbf{A}_7\}$ and $\{\mathbf{A}_3, \mathbf{A}_5, \mathbf{A}_6, \mathbf{A}_7\}$ are adjacent because all but one columns are the same. The corresponding basic solutions $\mathbf{x} = (0, 0, 0, 8, 12, 4, 6)$ and $\mathbf{x} = (0, 0, 4, 0, -12, 4, 6)$

are adjacent: we have $n = 7$ and a total of six common linearly independent active constraints; these are $x_1 \geq 0$, $x_2 \geq 0$, and the four equality constraints.

The full row rank assumption on A

We close this section by showing that the full row rank assumption on the matrix $\mathbf{A}$ results in no loss of generality.

Theorem 2.5 *Let $P = \{\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$ be a nonempty polyhedron, where $\mathbf{A}$ is a matrix of dimensions $m \times n$, with rows $\mathbf{a}_1', \ldots, \mathbf{a}_m'$. Suppose that $\text{rank}(\mathbf{A}) = k < m$ and that the rows $\mathbf{a}_{i_1}', \ldots, \mathbf{a}_{i_k}'$ are linearly independent. Consider the polyhedron*

$$Q = \{\mathbf{x} \mid \mathbf{a}_{i_1}'\mathbf{x} = b_{i_1}, \ldots, \mathbf{a}_{i_k}'\mathbf{x} = b_{i_k},\ \mathbf{x} \geq \mathbf{0}\}.$$

Then $Q = P$.

Proof. We provide the proof for the case where $i_1 = 1, \ldots, i_k = k$, that is, the first k rows of $\mathbf{A}$ are linearly independent. The general case can be reduced to this one by rearranging the rows of $\mathbf{A}$.

Clearly $P \subset Q$ since any element of P automatically satisfies the constraints defining Q. We will now show that $Q \subset P$.

Since $\text{rank}(\mathbf{A}) = k$, the row space of $\mathbf{A}$ has dimension k and the rows $\mathbf{a}_1', \ldots, \mathbf{a}_k'$ form a basis of the row space. Therefore, every row $\mathbf{a}_i'$ of $\mathbf{A}$ can be expressed in the form $\mathbf{a}_i' = \sum_{j=1}^{k} \lambda_{ij}\mathbf{a}_j'$, for some scalars λ_{ij}. Let $\mathbf{x}$ be an element of P and note that

$$b_i = \mathbf{a}_i'\mathbf{x} = \sum_{j=1}^{k} \lambda_{ij}\mathbf{a}_j'\mathbf{x} = \sum_{j=1}^{k} \lambda_{ij} b_j, \qquad i = 1, \ldots, m.$$

Consider now an element $\mathbf{y}$ of Q. We will show that it belongs to P. Indeed, for any i,

$$\mathbf{a}_i'\mathbf{y} = \sum_{j=1}^{k} \lambda_{ij}\mathbf{a}_j'\mathbf{y} = \sum_{j=1}^{k} \lambda_{ij} b_j = b_i,$$

which establishes that $\mathbf{y} \in P$ and $Q \subset P$. □

Notice that the polyhedron Q in Theorem 2.5 is in standard form; namely, $Q = \{\mathbf{x} \mid \mathbf{D}\mathbf{x} = \mathbf{f},\ \mathbf{x} \geq \mathbf{0}\}$ where $\mathbf{D}$ is a $k \times n$ submatrix of $\mathbf{A}$, with rank equal to k, and $\mathbf{f}$ is a k-dimensional subvector of $\mathbf{b}$. We conclude that as long as the feasible set is nonempty, a linear programming problem in standard form can be reduced to an equivalent standard form problem (with the same feasible set) in which the equality constraints are linearly independent.

Example 2.3 Consider the (nonempty) polyhedron defined by the constraints

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 2 \\ x_1 + x_2 \qquad &= 1 \\ x_1 \qquad + x_3 &= 1 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

The corresponding matrix $\mathbf{A}$ has rank two. This is because the last two rows $(1,1,0)$ and $(1,0,1)$ are linearly independent, but the first row is equal to the sum of the other two. Thus, the first constraint is redundant and after it is eliminated, we still have the same polyhedron.

2.4 Degeneracy

According to our definition, at a basic solution, we must have n linearly independent active constraints. This allows for the possibility that the number of active constraints is greater than n. (Of course, in n dimensions, no more than n of them can be linearly independent.) In this case, we say that we have a *degenerate* basic solution. In other words, at a degenerate basic solution, the number of active constraints is greater than the minimum necessary.

Definition 2.10 *A basic solution $\mathbf{x} \in \Re^n$ is said to be* **degenerate** *if more than n of the constraints are active at $\mathbf{x}$.*

In two dimensions, a degenerate basic solution is at the intersection of three or more lines; in three dimensions, a degenerate basic solution is at the intersection of four or more planes; see Figure 2.9 for an illustration. It turns out that the presence of degeneracy can strongly affect the behavior of linear programming algorithms and for this reason, we will now develop some more intuition.

Example 2.4 Consider the polyhedron P defined by the constraints

$$\begin{aligned} x_1 + x_2 + 2x_3 &\leq 8 \\ x_2 + 6x_3 &\leq 12 \\ x_1 &\leq 4 \\ x_2 &\leq 6 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

The vector $\mathbf{x} = (2,6,0)$ is a nondegenerate basic feasible solution, because there are exactly three active and linearly independent constraints, namely, $x_1 + x_2 + 2x_3 \leq 8$, $x_2 \leq 6$, and $x_3 \geq 0$. The vector $\mathbf{x} = (4,0,2)$ is a degenerate basic feasible solution, because there are four active constraints, three of them linearly independent, namely, $x_1 + x_2 + 2x_3 \leq 8$, $x_2 + 6x_3 \leq 12$, $x_1 \leq 4$, and $x_2 \geq 0$.

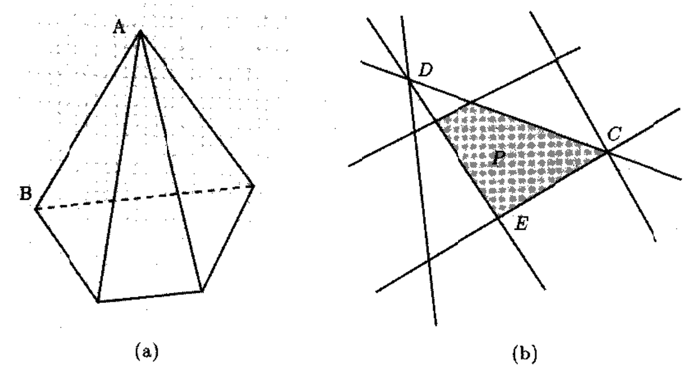

Figure 2.9: The points A and C are degenerate basic feasible solutions. The points B and E are nondegenerate basic feasible solutions. The point D is a degenerate basic solution.

Degeneracy in standard form polyhedra

At a basic solution of a polyhedron in standard form, the m equality constraints are always active. Therefore, having more than n active constraints is the same as having more than $n - m$ variables at zero level. This leads us to the next definition which is a special case of Definition 2.10.

Definition 2.11 *Consider the standard form polyhedron $P = \{\mathbf{x} \in \Re^n \mid \mathbf{Ax} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$ and let $\mathbf{x}$ be a basic solution. Let m be the number of rows of $\mathbf{A}$. The vector $\mathbf{x}$ is a* **degenerate** *basic solution if more than $n - m$ of the components of $\mathbf{x}$ are zero.*

Example 2.5 Consider once more the polyhedron of Example 2.4. By introducing the slack variables $x_4, \ldots, x_7$, we can transform it into the standard form $P = \{\mathbf{x} = (x_1, \ldots, x_7) \mid \mathbf{Ax} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 6 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 8 \\ 12 \\ 4 \\ 6 \end{bmatrix}.$$

Consider the basis consisting of the linearly independent columns $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, $\mathbf{A}_7$. To calculate the corresponding basic solution, we first set the nonbasic variables x_4, x_5, and x_6 to zero, and then solve the system $\mathbf{Ax} = \mathbf{b}$ for the remaining variables to obtain $\mathbf{x} = (4,0,2,0,0,0,6)$. This is a degenerate basic feasible solution, because we have a total of four variables that are zero, whereas

$n - m = 7 - 4 = 3$. Thus, while we initially set only the three nonbasic variables to zero, the solution to the system $\mathbf{Ax} = \mathbf{b}$ turned out to satisfy one more of the constraints (namely, the constraint $x_2 \geq 0$) with equality. Consider now the basis consisting of the linearly independent columns $\mathbf{A}_1$, $\mathbf{A}_3$, $\mathbf{A}_4$, and $\mathbf{A}_7$. The corresponding basic feasible solution is again $\mathbf{x} = (4, 0, 2, 0, 0, 0, 6)$.

The preceding example suggests that we can think of degeneracy in the following terms. We pick a basic solution by picking n linearly independent constraints to be satisfied with equality, and we realize that certain other constraints are also satisfied with equality. If the entries of $\mathbf{A}$ or $\mathbf{b}$ were chosen at random, this would almost never happen. Also, Figure 2.10 illustrates that if the coefficients of the active constraints are slightly perturbed, degeneracy can disappear (cf. Exercise 2.18). In practical problems, however, the entries of $\mathbf{A}$ and $\mathbf{b}$ often have a special (nonrandom) structure, and degeneracy is more common than the preceding argument would seem to suggest.

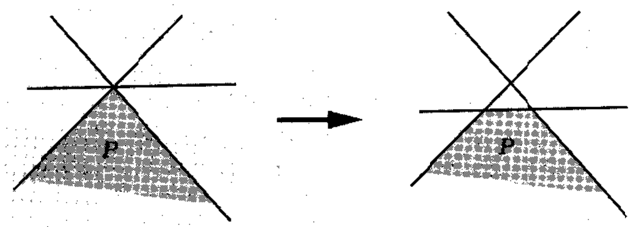

Figure 2.10: Small changes in the constraining inequalities can remove degeneracy.

In order to visualize degeneracy in standard form polyhedra, we assume that $n - m = 2$ and we draw the feasible set as a subset of the two-dimensional set defined by the equality constraints $\mathbf{Ax} = \mathbf{b}$; see Figure 2.11. At a nondegenerate basic solution, exactly $n-m$ of the constraints $x_i \geq 0$ are active; the corresponding variables are nonbasic. In the case of a degenerate basic solution, more than $n - m$ of the constraints $x_i \geq 0$ are active, and there are usually several ways of choosing which $n-m$ variables to call nonbasic; in that case, there are several bases corresponding to that same basic solution. (This discussion refers to the typical case. However, there are examples of degenerate basic solutions to which there corresponds only one basis.)

Degeneracy is not a purely geometric property

We close this section by pointing out that degeneracy of basic feasible solutions is not, in general, a geometric (representation independent) property,

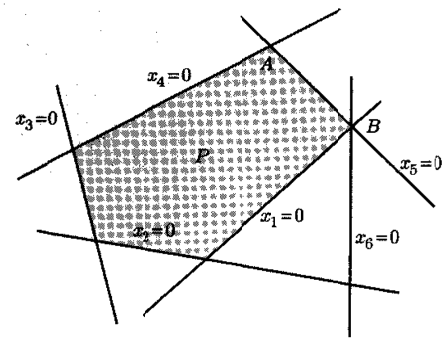

Figure 2.11: An $(n-m)$-dimensional illustration of degeneracy. Here, $n = 6$ and $m = 4$. The basic feasible solution A is nondegenerate and the basic variables are x_1, x_2, x_3, x_6. The basic feasible solution B is degenerate. We can choose x_1, x_6 as the nonbasic variables. Other possibilities are to choose x_1, x_5, or to choose x_5, x_6. Thus, there are three possible bases, for the same basic feasible solution B.

but rather it may depend on the particular representation of a polyhedron. To illustrate this point, consider the standard form polyhedron (cf. Figure 2.12)

$$P = \Big\{(x_1, x_2, x_3) \mid x_1 - x_2 = 0,\ x_1 + x_2 + 2x_3 = 2,\ x_1, x_2, x_3 \geq 0\Big\}.$$

We have $n = 3$, $m = 2$ and $n - m = 1$. The vector $(1, 1, 0)$ is nondegenerate because only one variable is zero. The vector $(0,0,1)$ is degenerate because two variables are zero. However, the same polyhedron can also be described

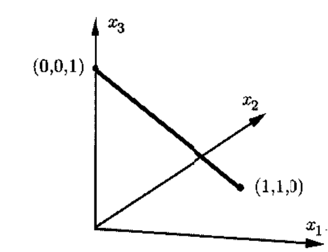

Figure 2.12: An example of degeneracy in a standard form problem.

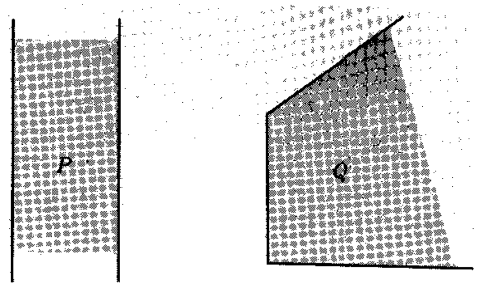

Figure 2.13: The polyhedron P contains a line and does not have an extreme point, while Q does not contain a line and has extreme points.

in the (nonstandard) form

$$P = \Big\{(x_1, x_2, x_3) \mid x_1 - x_2 = 0,\ x_1 + x_2 + 2x_3 = 2,\ x_1 \geq 0,\ x_3 \geq 0\Big\}.$$

The vector (0,0,1) is now a nondegenerate basic feasible solution, because there are only three active constraints.

For another example, consider a nondegenerate basic feasible solution $\mathbf{x}^*$ of a standard form polyhedron $P = \{\mathbf{x} \mid \mathbf{Ax} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$, where $\mathbf{A}$ is of dimensions $m \times n$. In particular, exactly $n - m$ of the variables x_i^* are equal to zero. Let us now represent P in the form $P = \{\mathbf{x} \mid \mathbf{Ax} \geq \mathbf{b},\ -\mathbf{Ax} \geq -\mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$. Then, at the basic feasible solution $\mathbf{x}^*$, we have $n - m$ variables set to zero and an additional $2m$ inequality constraints are satisfied with equality. We therefore have $n + m$ active constraints and $\mathbf{x}^*$ is degenerate. Hence, under the second representation, every basic feasible solution is degenerate.

We have established that a degenerate basic feasible solution under one representation could be nondegenerate under another representation. Still, it can be shown that if a basic feasible solution is degenerate under one particular standard form representation, then it is degenerate under every standard form representation of the same polyhedron (Exercise 2.19).

2.5 Existence of extreme points

We obtain in this section necessary and sufficient conditions for a polyhedron to have at least one extreme point. We first observe that not every polyhedron has this property. For example, if $n > 1$, a halfspace in $\Re^n$ is a polyhedron without extreme points. Also, as argued in Section 2.2 (cf. the discussion after Definition 2.9), if the matrix $\mathbf{A}$ has fewer than n rows, then the polyhedron $\{\mathbf{x} \in \Re^n \mid \mathbf{Ax} \geq \mathbf{b}\}$ does not have a basic feasible solution.

It turns out that the existence of an extreme point depends on whether a polyhedron contains an infinite line or not; see Figure 2.13. We need the following definition.

Definition 2.12 *A polyhedron $P \subset \Re^n$* **contains a line** *if there exists a vector $\mathbf{x} \in P$ and a nonzero vector $\mathbf{d} \in \Re^n$ such that $\mathbf{x} + \lambda\mathbf{d} \in P$ for all scalars λ.*

We then have the following result.

Theorem 2.6 *Suppose that the polyhedron $P = \{\mathbf{x} \in \Re^n \mid \mathbf{a}_i'\mathbf{x} \geq b_i,\ i = 1, \ldots, m\}$ is nonempty. Then, the following are equivalent:*

(a) *The polyhedron P has at least one extreme point.*

(b) *The polyhedron P does not contain a line.*

(c) *There exist n vectors out of the family $\mathbf{a}_1, \ldots, \mathbf{a}_m$, which are linearly independent.*

Proof.

(b) ⇒ (a)

We first prove that if P does not contain a line, then it has a basic feasible solution and, therefore, an extreme point. A geometric interpretation of this proof is provided in Figure 2.14.

Let $\mathbf{x}$ be an element of P and let $I = \{i \mid \mathbf{a}_i'\mathbf{x} = b_i\}$. If n of the vectors $\mathbf{a}_i$, $i \in I$, corresponding to the active constraints are linearly independent, then $\mathbf{x}$ is, by definition, a basic feasible solution and, therefore, a basic feasible solution exists. If this is not the case, then all of the vectors $\mathbf{a}_i$, $i \in I$, lie in a proper subspace of $\Re^n$ and there exists a nonzero vector $\mathbf{d} \in \Re^n$ such that $\mathbf{a}_i'\mathbf{d} = 0$, for every $i \in I$. Let us consider the line consisting of all points of the form $\mathbf{y} = \mathbf{x} + \lambda\mathbf{d}$, where λ is an arbitrary scalar. For $i \in I$, we have $\mathbf{a}_i'\mathbf{y} = \mathbf{a}_i'\mathbf{x} + \lambda\mathbf{a}_i'\mathbf{d} = \mathbf{a}_i'\mathbf{x} = b_i$. Thus, those constraints that were active at $\mathbf{x}$ remain active at all points on the line. However, since the polyhedron is assumed to contain no lines, it follows that as we vary λ, some constraint will be eventually violated. At the point where some constraint is about to be violated, a new constraint must become active, and we conclude that there exists some λ^* and some $j \notin I$ such that $\mathbf{a}_j'(\mathbf{x} + \lambda^*\mathbf{d}) = b_j$.

We claim that $\mathbf{a}_j$ is not a linear combination of the vectors $\mathbf{a}_i$, $i \in I$. Indeed, we have $\mathbf{a}_j'\mathbf{x} \neq b_j$ (because $j \notin I$) and $\mathbf{a}_j'(\mathbf{x} + \lambda^*\mathbf{d}) = b_j$ (by the definition of λ^*). Thus, $\mathbf{a}_j'\mathbf{d} \neq 0$. On the other hand, $\mathbf{a}_i'\mathbf{d} = 0$ for every $i \in I$ (by the definition of $\mathbf{d}$) and therefore, $\mathbf{d}$ is orthogonal to any linear combination of the vectors $\mathbf{a}_i$, $i \in I$. Since $\mathbf{d}$ is not orthogonal to $\mathbf{a}_j$, we

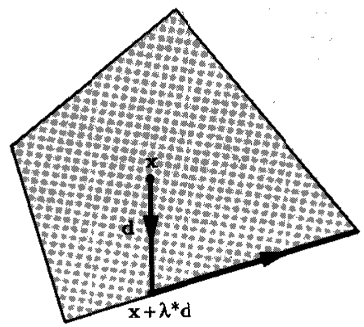

Figure 2.14: Starting from an arbitrary point of a polyhedron, we choose a direction along which all currently active constraints remain active. We then move along that direction until a new constraint is about to be violated. At that point, the number of linearly independent active constraints has increased by at least one. We repeat this procedure until we end up with n linearly independent active constraints, at which point we have a basic feasible solution.

conclude that $\mathbf{a}_j$ is a not a linear combination of the vectors $\mathbf{a}_i$, $i \in I$. Thus, by moving from $\mathbf{x}$ to $\mathbf{x} + \lambda^* \mathbf{d}$, the number of linearly independent active constraints has been increased by at least one. By repeating the same argument, as many times as needed, we eventually end up with a point at which there are n linearly independent active constraints. Such a point is, by definition, a basic solution; it is also feasible since we have stayed within the feasible set.

(a) ⇒ (c)

If P has an extreme point $\mathbf{x}$, then $\mathbf{x}$ is also a basic feasible solution (cf. Theorem 2.3), and there exist n constraints that are active at $\mathbf{x}$, with the corresponding vectors $\mathbf{a}_i$ being linearly independent.

(c) ⇒ (b)

Suppose that n of the vectors $\mathbf{a}_i$ are linearly independent and, without loss of generality, let us assume that $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are linearly independent. Suppose that P contains a line $\mathbf{x} + \lambda \mathbf{d}$, where $\mathbf{d}$ is a nonzero vector. We then have $\mathbf{a}_i'(\mathbf{x} + \lambda \mathbf{d}) \geq b_i$ for all i and all λ. We conclude that $\mathbf{a}_i'\mathbf{d} = 0$ for all i. (If $\mathbf{a}_i'\mathbf{d} < 0$, we can violate the constraint by picking λ very large; a symmetric argument applies if $\mathbf{a}_i'\mathbf{d} > 0$.) Since the vectors $\mathbf{a}_i$, $i = 1, \ldots, n$, are linearly independent, this implies that $\mathbf{d} = \mathbf{0}$. This is a contradiction and establishes that P does not contain a line. □

Notice that a bounded polyhedron does not contain a line. Similarly, the positive orthant $\{\mathbf{x} \mid \mathbf{x} \geq \mathbf{0}\}$ does not contain a line. Since a polyhedron in standard form is contained in the positive orthant, it does not contain a line either. These observations establish the following important corollary of Theorem 2.6.

Corollary 2.2 *Every nonempty bounded polyhedron and every nonempty polyhedron in standard form has at least one basic feasible solution.*

2.6 Optimality of extreme points

Having established the conditions for the existence of extreme points, we will now confirm the intuition developed in Chapter 1: as long as a linear programming problem has an optimal solution and as long as the feasible set has at least one extreme point, we can always find an optimal solution within the set of extreme points of the feasible set. Later in this section, we prove a somewhat stronger result, at the expense of a more complicated proof.

Theorem 2.7 *Consider the linear programming problem of minimizing $\mathbf{c}'\mathbf{x}$ over a polyhedron P. Suppose that P has at least one extreme point and that there exists an optimal solution. Then, there exists an optimal solution which is an extreme point of P.*

Proof. (See Figure 2.15 for an illustration.) Let Q be the set of all optimal solutions, which we have assumed to be nonempty. Let P be of the form $P = \{\mathbf{x} \in \Re^n \mid \mathbf{A}\mathbf{x} \geq \mathbf{b}\}$ and let v be the optimal value of the cost $\mathbf{c}'\mathbf{x}$. Then, $Q = \{\mathbf{x} \in \Re^n \mid \mathbf{A}\mathbf{x} \geq \mathbf{b},\ \mathbf{c}'\mathbf{x} = v\}$, which is also a polyhedron. Since

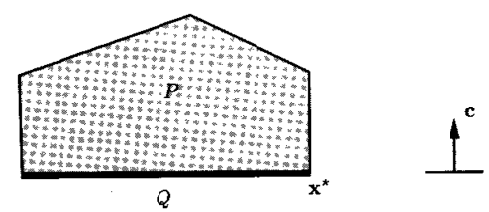

Figure 2.15: Illustration of the proof of Theorem 2.7. Here, Q is the set of optimal solutions and an extreme point $\mathbf{x}^*$ of Q is also an extreme point of P.

$Q \subset P$, and since P contains no lines (cf. Theorem 2.6), Q contains no lines either. Therefore, Q has an extreme point.

Let $\mathbf{x}^*$ be an extreme point of Q. We will show that $\mathbf{x}^*$ is also an extreme point of P. Suppose, in order to derive a contradiction, that $\mathbf{x}^*$ is not an extreme point of P. Then, there exist $\mathbf{y} \in P$, $\mathbf{z} \in P$, such that $\mathbf{y} \neq \mathbf{x}^*$, $\mathbf{z} \neq \mathbf{x}^*$, and some $\lambda \in [0,1]$ such that $\mathbf{x}^* = \lambda\mathbf{y} + (1-\lambda)\mathbf{z}$. It follows that $v = \mathbf{c}'\mathbf{x}^* = \lambda\mathbf{c}'\mathbf{y} + (1-\lambda)\mathbf{c}'\mathbf{z}$. Furthermore, since v is the optimal cost, $\mathbf{c}'\mathbf{y} \geq v$ and $\mathbf{c}'\mathbf{z} \geq v$. This implies that $\mathbf{c}'\mathbf{y} = \mathbf{c}'\mathbf{z} = v$ and therefore $\mathbf{z} \in Q$ and $\mathbf{y} \in Q$. But this contradicts the fact that $\mathbf{x}^*$ is an extreme point of Q. The contradiction establishes that $\mathbf{x}^*$ is an extreme point of P. In addition, since $\mathbf{x}^*$ belongs to Q, it is optimal. □

The above theorem applies to polyhedra in standard form, as well as to bounded polyhedra, since they do not contain a line.

Our next result is stronger than Theorem 2.7. It shows that the existence of an optimal solution can be taken for granted, as long as the optimal cost is finite.

> **Theorem 2.8** *Consider the linear programming problem of minimizing $\mathbf{c}'\mathbf{x}$ over a polyhedron P. Suppose that P has at least one extreme point. Then, either the optimal cost is equal to $-\infty$, or there exists an extreme point which is optimal.*

Proof. The proof is essentially a repetition of the proof of Theorem 2.6. The difference is that as we move towards a basic feasible solution, we will also make sure that the costs do not increase. We will use the following terminology: an element $\mathbf{x}$ of P has *rank* k if we can find k, but not more than k, linearly independent constraints that are active at $\mathbf{x}$.

Let us assume that the optimal cost is finite. Let $P = \{\mathbf{x} \in \Re^n \mid \mathbf{A}\mathbf{x} \geq \mathbf{b}\}$ and consider some $\mathbf{x} \in P$ of rank $k < n$. We will show that there exists some $\mathbf{y} \in P$ which has greater rank and satisfies $\mathbf{c}'\mathbf{y} \leq \mathbf{c}'\mathbf{x}$. Let $I = \{i \mid \mathbf{a}_i'\mathbf{x} = b_i\}$, where $\mathbf{a}_i'$ is the ith row of $\mathbf{A}$. Since $k < n$, the vectors $\mathbf{a}_i$, $i \in I$, lie in a proper subspace of $\Re^n$, and we can choose some nonzero $\mathbf{d} \in \Re^n$ orthogonal to every $\mathbf{a}_i$, $i \in I$. Furthermore, by possibly taking the negative of $\mathbf{d}$, we can assume that $\mathbf{c}'\mathbf{d} \leq \mathbf{0}$.

Suppose that $\mathbf{c}'\mathbf{d} < 0$. Let us consider the half-line $\mathbf{y} = \mathbf{x} + \lambda\mathbf{d}$, where λ is a positive scalar. As in the proof of Theorem 2.6, all points on this half-line satisfy the relations $\mathbf{a}_i'\mathbf{y} = b_i$, $i \in I$. If the entire half-line were contained in P, the optimal cost would be $-\infty$, which we have assumed not to be the case. Therefore, the half-line eventually exits P. When this is about to happen, we have some $\lambda^* > 0$ and $j \notin I$ such that $\mathbf{a}_j'(\mathbf{x} + \lambda^*\mathbf{d}) = b_j$. We let $\mathbf{y} = \mathbf{x} + \lambda^*\mathbf{d}$ and note that $\mathbf{c}'\mathbf{y} < \mathbf{c}'\mathbf{x}$. As in the proof of Theorem 2.6, $\mathbf{a}_j$ is linearly independent from $\mathbf{a}_i$, $i \in I$, and the rank of $\mathbf{y}$ is at least $k+1$.

Suppose now that $\mathbf{c}'\mathbf{d} = 0$. We consider the line $\mathbf{y} = \mathbf{x} + \lambda\mathbf{d}$, where λ is an arbitrary scalar. Since P contains no lines, the line must eventually exit P and when that is about to happen, we are again at a vector $\mathbf{y}$ of rank greater than that of $\mathbf{x}$. Furthermore, since $\mathbf{c}'\mathbf{d} = 0$, we have $\mathbf{c}'\mathbf{y} = \mathbf{c}'\mathbf{x}$.

In either case, we have found a new point $\mathbf{y}$ such that $\mathbf{c}'\mathbf{y} \leq \mathbf{c}'\mathbf{x}$, and whose rank is greater than that of $\mathbf{x}$. By repeating this process as many times as needed, we end up with a vector $\mathbf{w}$ of rank n (thus, $\mathbf{w}$ is a basic feasible solution) such that $\mathbf{c}'\mathbf{w} \leq \mathbf{c}'\mathbf{x}$.

Let $\mathbf{w}^1, \ldots, \mathbf{w}^r$ be the basic feasible solutions in P and let $\mathbf{w}^*$ be a basic feasible solution such that $\mathbf{c}'\mathbf{w}^* \leq \mathbf{c}'\mathbf{w}^i$ for all i. We have already shown that for every $\mathbf{x}$ there exists some i such that $\mathbf{c}'\mathbf{w}^i \leq \mathbf{c}'\mathbf{x}$. It follows that $\mathbf{c}'\mathbf{w}^* \leq \mathbf{c}'\mathbf{x}$ for all $\mathbf{x} \in P$, and the basic feasible solution $\mathbf{w}^*$ is optimal. □

For a general linear programming problem, if the feasible set has no extreme points, then Theorem 2.8 does not apply directly. On the other hand, any linear programming problem can be transformed into an equivalent problem in standard form to which Theorem 2.8 does apply. This establishes the following corollary.

> **Corollary 2.3** *Consider the linear programming problem of minimizing $\mathbf{c}'\mathbf{x}$ over a nonempty polyhedron. Then, either the optimal cost is equal to $-\infty$ or there exists an optimal solution.*

The result in Corollary 2.3 should be contrasted with what may happen in optimization problems with a nonlinear cost function. For example, in the problem of minimizing $1/x$ subject to $x \geq 1$, the optimal cost is not $-\infty$, but an optimal solution does not exist.

2.7 Representation of bounded polyhedra*

So far, we have been representing polyhedra in terms of their defining inequalities. In this section, we provide an alternative, by showing that a bounded polyhedron can also be represented as the convex hull of its extreme points. The proof that we give here is elementary and constructive, and its main idea is summarized in Figure 2.16. There is a similar representation of unbounded polyhedra involving extreme points and "extreme rays" (edges that extend to infinity). This representation can be developed using the tools that we already have, at the expense of a more complicated proof. A more elegant argument, based on duality theory, will be presented in Section 4.9 and will also result in an alternative proof of Theorem 2.9 below.

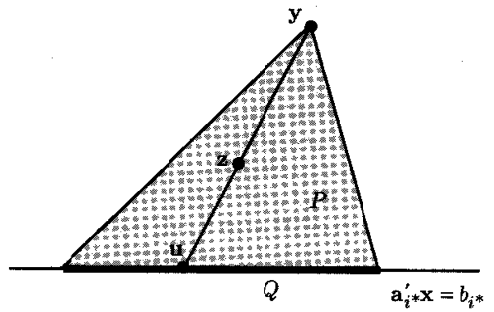

Figure 2.16: Given the vector $\mathbf{z}$, we express it as a convex combination of $\mathbf{y}$ and $\mathbf{u}$. The vector $\mathbf{u}$ belongs to the polyhedron Q whose dimension is lower than that of P. Using induction on dimension, we can express the vector $\mathbf{u}$ as a convex combination of extreme points of Q. These are also extreme points of P.

> **Theorem 2.9** *A nonempty and bounded polyhedron is the convex hull of its extreme points.*

Proof. Every convex combination of extreme points is an element of the polyhedron, since polyhedra are convex sets. Thus, we only need to prove the converse result and show that every element of a bounded polyhedron can be represented as a convex combination of extreme points.

We define the *dimension* of a polyhedron $P \subset \Re^n$ as the smallest integer k such that P is contained in some k-dimensional affine subspace of $\Re^n$. (Recall from Section 1.5, that a k-dimensional affine subspace is a translation of a k-dimensional subspace.) Our proof proceeds by induction on the dimension of the polyhedron P. If P is zero-dimensional, it consists of a single point. This point is an extreme point of P and the result is true.

Let us assume that the result is true for all polyhedra of dimension less than k. Let $P = \{\mathbf{x} \in \Re^n \mid \mathbf{a}_i'\mathbf{x} \geq b_i,\ i = 1, \ldots, m\}$ be a nonempty bounded k-dimensional polyhedron. Then, P is contained in a k-dimensional affine subspace S of $\Re^n$, which can be assumed to be of the form

$$S = \{\mathbf{x}^0 + \lambda_1\mathbf{x}^1 + \cdots + \lambda_k\mathbf{x}^k \mid \lambda_1, \ldots, \lambda_k \in \Re\},$$

where $\mathbf{x}^1, \ldots, \mathbf{x}^k$ are some vectors in $\Re^n$. Let $\mathbf{f}_1, \ldots, \mathbf{f}_{n-k}$ be $n-k$ linearly independent vectors that are orthogonal to $\mathbf{x}^1, \ldots, \mathbf{x}^k$. Let $g_i = \mathbf{f}_i'\mathbf{x}^0$, for $i = 1, \ldots, n-k$. Then, every element $\mathbf{x}$ of S satisfies

$$\mathbf{f}_i'\mathbf{x} = g_i, \qquad i = 1, \ldots, n-k. \tag{2.3}$$

Since $P \subset S$, the same must be true for every element of P.

Let $\mathbf{z}$ be an element of P. If $\mathbf{z}$ is an extreme point of P, then $\mathbf{z}$ is a trivial convex combination of the extreme points of P and there is nothing more to be proved. If $\mathbf{z}$ is not an extreme point of P, let us choose an arbitrary extreme point $\mathbf{y}$ of P and form the half-line consisting of all points of the form $\mathbf{z} + \lambda(\mathbf{z} - \mathbf{y})$, where λ is a nonnegative scalar. Since P is bounded, this half-line must eventually exit P and violate one of the constraints, say the constraint $\mathbf{a}_{i^*}'\mathbf{x} \geq b_{i^*}$. By considering what happens when this constraint is just about to be violated, we find some $\lambda^* \geq 0$ and $\mathbf{u} \in P$, such that

$$\mathbf{u} = \mathbf{z} + \lambda^*(\mathbf{z} - \mathbf{y}),$$

and

$$\mathbf{a}_{i^*}'\mathbf{u} = b_{i^*}.$$

Since the constraint $\mathbf{a}_{i^*}'\mathbf{x} \geq b_{i^*}$ is violated if λ grows beyond λ^*, it follows that $\mathbf{a}_{i^*}'(\mathbf{z} - \mathbf{y}) < 0$.

Let Q be the polyhedron defined by

$$\begin{aligned} Q &= \{\mathbf{x} \in P \mid \mathbf{a}_{i^*}'\mathbf{x} = b_{i^*}\} \\ &= \{\mathbf{x} \in \Re^n \mid \mathbf{a}_i'\mathbf{x} \geq b_i,\ i = 1, \ldots, m,\ \mathbf{a}_{i^*}'\mathbf{x} = b_{i^*}\}. \end{aligned}$$

Since $\mathbf{z}, \mathbf{y} \in P$, we have $\mathbf{f}_i'\mathbf{z} = g_i = \mathbf{f}_i'\mathbf{y}$ which shows that $\mathbf{z} - \mathbf{y}$ is orthogonal to each vector $\mathbf{f}_i$, for $i = 1, \ldots, n-k$. On the other hand, we have shown that $\mathbf{a}_{i^*}'(\mathbf{z} - \mathbf{y}) < 0$, which implies that the vector $\mathbf{a}_{i^*}$ is not a linear combination of, and is therefore linearly independent from, the vectors $\mathbf{f}_i$. Note that

$$Q \subset \{\mathbf{x} \in \Re^n \mid \mathbf{a}_{i^*}'\mathbf{x} = b_{i^*},\ \mathbf{f}_i'\mathbf{x} = g_i,\ i = 1, \ldots, n-k\},$$

since Eq. (2.3) holds for every element of P. The set on the right is defined by $n-k+1$ linearly independent equality constraints. Hence, it is an affine subspace of dimension $k-1$ (see the discussion at the end of Section 1.5). Therefore, Q has dimension at most $k-1$.

By applying the induction hypothesis to Q and $\mathbf{u}$, we see that $\mathbf{u}$ can be expressed as a convex combination

$$\mathbf{u} = \sum_i \lambda_i \mathbf{v}^i$$

of the extreme points $\mathbf{v}^i$ of Q, where λ_i are nonnegative scalars that sum to one. Note that at an extreme point $\mathbf{v}$ of Q, we must have $\mathbf{a}_i'\mathbf{v} = b_i$ for n linearly independent vectors $\mathbf{a}_i$; therefore, $\mathbf{v}$ must also be an extreme point of P. Using the definition of λ^*, we also have

$$\mathbf{z} = \frac{\mathbf{u} + \lambda^*\mathbf{y}}{1 + \lambda^*}.$$

Therefore,

$$\mathbf{z} = \frac{\lambda^* \mathbf{y}}{1+\lambda^*} + \sum_i \frac{\lambda_i}{1+\lambda^*}\mathbf{v}^i,$$

which shows that $\mathbf{z}$ is a convex combination of the extreme points of P. □

Example 2.6 Consider the polyhedron

$$P = \{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 \le 1,\ x_1, x_2, x_3 \ge 0\}.$$

It has four extreme points, namely, $\mathbf{x}^1 = (1,0,0)$, $\mathbf{x}^2 = (0,1,0)$, $\mathbf{x}^3 = (0,0,1)$, and $\mathbf{x}^4 = (0,0,0)$. The vector $\mathbf{x} = (1/3, 1/3, 1/4)$ belongs to P. It can be represented as

$$\mathbf{x} = \frac{1}{3}\mathbf{x}^1 + \frac{1}{3}\mathbf{x}^2 + \frac{1}{4}\mathbf{x}^3 + \frac{1}{12}\mathbf{x}^4.$$

There is a converse to Theorem 2.9 asserting that the convex hull of a finite number of points is a polyhedron. This result is proved in the next section and again in Section 4.9.

2.8 Projections of polyhedra: Fourier-Motzkin elimination*

In this section, we present perhaps the oldest method for solving linear programming problems. This method is not practical because it requires a very large number of steps, but it has some interesting theoretical corollaries.

The key to this method is the concept of a *projection*, defined as follows: if $\mathbf{x} = (x_1, \ldots, x_n)$ is a vector in $\Re^n$ and $k \le n$, the projection mapping $\pi_k : \Re^n \mapsto \Re^k$ projects $\mathbf{x}$ onto its first k coordinates:

$$\pi_k(\mathbf{x}) = \pi_k(x_1, \ldots, x_n) = (x_1, \ldots, x_k).$$

We also define the projection $\Pi_k(S)$ of a set $S \subset \Re^n$ by letting

$$\Pi_k(S) = \{\pi_k(\mathbf{x}) \mid \mathbf{x} \in S\};$$

see Figure 2.17 for an illustration. Note that S is nonempty if and only if $\Pi_k(S)$ is nonempty. An equivalent definition is

$$\Pi_k(S) = \Big\{(x_1, \ldots, x_k) \mid \text{there exist } x_{k+1}, \ldots, x_n \text{ s.t. } (x_1, \ldots, x_n) \in S\Big\}.$$

Suppose now that we wish to decide whether a given polyhedron $P \subset \Re^n$ is nonempty. If we can somehow eliminate the variable x_n and construct the set $\Pi_{n-1}(P) \subset \Re^{n-1}$, we can instead consider the presumably easier problem of deciding whether $\Pi_{n-1}(P)$ is nonempty. If we keep eliminating variables one by one, we eventually arrive at the set $\Pi_1(P)$ that

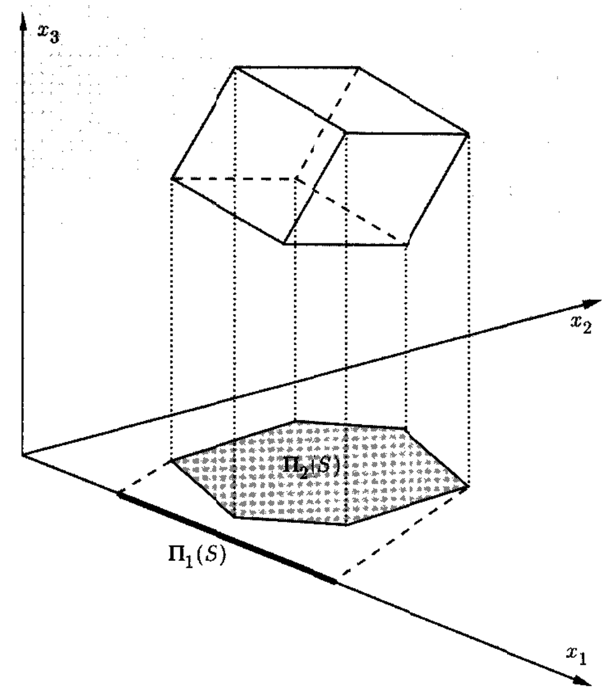

Figure 2.17: The projections $\Pi_1(S)$ and $\Pi_2(S)$ of a rotated three-dimensional cube.

involves a single variable, and whose emptiness is easy to check. The main disadvantage of this method is that while each step reduces the dimension by one, a large number of constraints is usually added. Exercise 2.20 deals with a family of examples in which the number of constraints increases exponentially with the problem dimension.

We now describe the elimination method. We are given a polyhedron P in terms of linear inequality constraints of the form

$$\sum_{j=1}^{n} a_{ij}x_j \ge b_i, \qquad i = 1, \ldots, m.$$

We wish to eliminate x_n and construct the projection $\Pi_{n-1}(P)$.

Elimination algorithm

1. Rewrite each constraint $\sum_{j=1}^{n} a_{ij}x_j \geq b_i$ in the form

$$a_{in}x_n \geq -\sum_{j=1}^{n-1} a_{ij}x_j + b_i, \qquad i = 1,\ldots,m;$$

if $a_{in} \neq 0$, divide both sides by a_{in}. By letting $\overline{\mathbf{x}} = (x_1,\ldots,x_{n-1})$, we obtain an equivalent representation of P involving the following constraints:

$$x_n \geq d_i + \mathbf{f}_i'\overline{\mathbf{x}}, \qquad \text{if } a_{in} > 0, \tag{2.4}$$
$$d_j + \mathbf{f}_j'\overline{\mathbf{x}} \geq x_n, \qquad \text{if } a_{jn} < 0, \tag{2.5}$$
$$0 \geq d_k + \mathbf{f}_k'\overline{\mathbf{x}}, \qquad \text{if } a_{kn} = 0. \tag{2.6}$$

Here, each d_i, d_j, d_k is a scalar, and each $\mathbf{f}_i$, $\mathbf{f}_j$, $\mathbf{f}_k$ is a vector in $\Re^{n-1}$.

2. Let Q be the polyhedron in $\Re^{n-1}$ defined by the constraints

$$d_j + \mathbf{f}_j'\overline{\mathbf{x}} \geq d_i + \mathbf{f}_i'\overline{\mathbf{x}}, \qquad \text{if } a_{in} > 0 \text{ and } a_{jn} < 0, \tag{2.7}$$
$$0 \geq d_k + \mathbf{f}_k'\overline{\mathbf{x}}, \qquad \text{if } a_{kn} = 0. \tag{2.8}$$

Example 2.7 Consider the polyhedron defined by the constraints

$$\begin{aligned} x_1 + x_2 &\geq 1 \\ x_1 + x_2 + 2x_3 &\geq 2 \\ 2x_1 + 3x_3 &\geq 3 \\ x_1 - 4x_3 &\geq 4 \\ -2x_1 + x_2 - x_3 &\geq 5. \end{aligned}$$

We rewrite these constraints in the form

$$\begin{aligned} 0 &\geq 1 - x_1 - x_2 \\ x_3 &\geq 1 - (x_1/2) - (x_2/2) \\ x_3 &\geq 1 - (2x_1/3) \\ -1 + (x_1/4) &\geq x_3 \\ -5 - 2x_1 + x_2 &\geq x_3. \end{aligned}$$

Then, the set Q is defined by the constraints

$$\begin{aligned} 0 &\geq 1 - x_1 - x_2 \\ -1 + x_1/4 &\geq 1 - (x_1/2) - (x_2/2) \\ -1 + x_1/4 &\geq 1 - (2x_1/3) \\ -5 - 2x_1 + x_2 &\geq 1 - (x_1/2) - (x_2/2) \\ -5 - 2x_1 + x_2 &\geq 1 - (2x_1/3). \end{aligned}$$

Theorem 2.10 *The polyhedron Q constructed by the elimination algorithm is equal to the projection $\Pi_{n-1}(P)$ of P.*

Proof. If $\overline{\mathbf{x}} \in \Pi_{n-1}(P)$, there exists some x_n such that $(\overline{\mathbf{x}}, x_n) \in P$. In particular, the vector $\mathbf{x} = (\overline{\mathbf{x}}, x_n)$ satisfies Eqs. (2.4)-(2.6), from which it follows immediately that $\overline{\mathbf{x}}$ satisfies Eqs. (2.7)-(2.8), and $\overline{\mathbf{x}} \in Q$. This shows that $\Pi_{n-1}(P) \subset Q$.

We will now prove that $Q \subset \Pi_{n-1}(P)$. Let $\overline{\mathbf{x}} \in Q$. It follows from Eq. (2.7) that

$$\min_{\{j \mid a_{jn}<0\}} (d_j + \mathbf{f}_j'\overline{\mathbf{x}}) \geq \max_{\{i \mid a_{in}>0\}} (d_i + \mathbf{f}_i'\overline{\mathbf{x}}).$$

Let x_n be any number between the two sides of the above inequality. It then follows that $(\overline{\mathbf{x}}, x_n)$ satisfies Eqs. (2.4)-(2.6) and, therefore, belongs to the polyhedron P. □

Notice that for any vector $\mathbf{x} = (x_1,\ldots,x_n)$, we have

$$\pi_{n-2}(\pi_{n-1}(\mathbf{x})) = (x_1,\ldots,x_{n-2}) = \pi_{n-2}(\mathbf{x}).$$

Accordingly, for any polyhedron P, we also have

$$\Pi_{n-2}(\Pi_{n-1}(P)) = \Pi_{n-2}(P).$$

By generalizing this observation, we see that if we apply the elimination algorithm k times, we end up with the set $\Pi_{n-k}(P)$; if we apply it $n-1$ times, we end up with $\Pi_1(P)$. Unfortunately, each application of the elimination algorithm can increase the number of constraints substantially, leading to a polyhedron $\Pi_1(P)$ described by a very large number of constraints. Of course, since $\Pi_1(P)$ is one-dimensional, almost all of these constraints will be redundant, but this is of no help: in order to decide which ones are redundant, we must, in general, enumerate them.

The elimination algorithm has an important theoretical consequence: since the projection $\Pi_k(P)$ can be generated by repeated application of the elimination algorithm, and since the elimination algorithm always produces a polyhedron, it follows that a projection $\Pi_k(P)$ of a polyhedron is also a polyhedron. This fact might be considered obvious, but a proof simpler than the one we gave is not apparent. We now restate it in somewhat different language.

Corollary 2.4 *Let $P \subset \Re^{n+k}$ be a polyhedron. Then, the set*

$$\{\mathbf{x} \in \Re^n \mid \text{there exists } \mathbf{y} \in \Re^k \text{ such that } (\mathbf{x}, \mathbf{y}) \in P\}$$

is also a polyhedron.

A variation of Corollary 2.4 states that the image of a polyhedron under a linear mapping is also a polyhedron.

Corollary 2.5 *Let $P \subset \Re^n$ be a polyhedron and let $\mathbf{A}$ be an $m \times n$ matrix. Then, the set $Q = \{\mathbf{A}\mathbf{x} \mid \mathbf{x} \in P\}$ is also a polyhedron.*

Proof. We have $Q = \{\mathbf{y} \in \Re^m \mid \text{there exists } \mathbf{x} \in \Re^n \text{ such that } \mathbf{A}\mathbf{x} = \mathbf{y},\ \mathbf{x} \in P\}$. Therefore, Q is the projection of the polyhedron $\{(\mathbf{x}, \mathbf{y}) \in \Re^{n+m} \mid \mathbf{A}\mathbf{x} = \mathbf{y},\ \mathbf{x} \in P\}$ onto the $\mathbf{y}$ coordinates. □

Corollary 2.6 *The convex hull of a finite number of vectors is a polyhedron.*

Proof. The convex hull

$$\left\{ \sum_{i=1}^{k} \lambda_i \mathbf{x}^i \;\middle|\; \sum_{i=1}^{k} \lambda_i = 1,\ \lambda_i \geq 0 \right\}$$

of a finite number of vectors $\mathbf{x}^1, \ldots, \mathbf{x}^k$ is the image of the polyhedron

$$\left\{ (\lambda_1, \ldots, \lambda_k) \;\middle|\; \sum_{i=1}^{k} \lambda_i = 1,\ \lambda_i \geq 0 \right\}$$

under the linear mapping that maps $(\lambda_1, \ldots, \lambda_k)$ to $\sum_{i=1}^{k} \lambda_i \mathbf{x}^i$ and is, therefore, a polyhedron. □

We finally indicate how the elimination algorithm can be used to solve linear programming problems. Consider the problem of minimizing $\mathbf{c}'\mathbf{x}$ subject to $\mathbf{x}$ belonging to a polyhedron P. We define a new variable x_0 and introduce the constraint $x_0 = \mathbf{c}'\mathbf{x}$. If we use the elimination algorithm n times to eliminate the variables $x_1, \ldots, x_n$, we are left with the set

$$Q = \{x_0 \mid \text{there exists } \mathbf{x} \in P \text{ such that } x_0 = \mathbf{c}'\mathbf{x}\},$$

and the optimal cost is equal to the smallest element of Q. An optimal solution $\mathbf{x}$ can be recovered by backtracking (Exercise 2.21).

2.9 Summary

We summarize our main conclusions so far regarding the solutions to linear programming problems.

(a) If the feasible set is nonempty and bounded, there exists an optimal solution. Furthermore, there exists an optimal solution which is an extreme point.

(b) If the feasible set is unbounded, there are the following possibilities:
   - (i) There exists an optimal solution which is an extreme point.
   - (ii) There exists an optimal solution, but no optimal solution is an extreme point. (This can only happen if the feasible set has no extreme points; it never happens when the problem is in standard form.)
   - (iii) The optimal cost is $-\infty$.

Suppose now that the optimal cost is finite and that the feasible set contains at least one extreme point. Since there are only finitely many extreme points, the problem can be solved in a finite number of steps, by enumerating all extreme points and evaluating the cost of each one. This is hardly a practical algorithm because the number of extreme points can increase exponentially with the number of variables and constraints. In the next chapter, we will exploit the geometry of the feasible set and develop the *simplex method*, a systematic procedure that moves from one extreme point to another, without having to enumerate all extreme points.

An interesting aspect of the material in this chapter is the distinction between geometric (representation independent) properties of a polyhedron and those properties that depend on a particular representation. In that respect, we have established the following:

(a) Whether or not a point is an extreme point (equivalently, vertex, or basic feasible solution) is a geometric property.

(b) Whether or not a point is a basic solution may depend on the way that a polyhedron is represented.

(c) Whether or not a basic or basic feasible solution is degenerate may depend on the way that a polyhedron is represented.

2.10 Exercises

Exercise 2.1 For each one of the following sets, determine whether it is a polyhedron.

(a) The set of all $(x, y) \in \Re^2$ satisfying the constraints

$$\begin{aligned} x\cos\theta + y\sin\theta &\leq 1, \qquad \forall\, \theta \in [0, \pi/2], \\ x &\geq 0, \\ y &\geq 0. \end{aligned}$$

(b) The set of all $x \in \Re$ satisfying the constraint $x^2 - 8x + 15 \leq 0$.

(c) The empty set.

Exercise 2.2 Let $f : \Re^n \mapsto \Re$ be a convex function and let c be some constant. Show that the set $S = \{\mathbf{x} \in \Re^n \mid f(\mathbf{x}) \leq c\}$ is convex.

Exercise 2.3 (Basic feasible solutions in standard form polyhedra with upper bounds) Consider a polyhedron defined by the constraints $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{0} \leq \mathbf{x} \leq \mathbf{u}$, and assume that the matrix $\mathbf{A}$ has linearly independent rows. Provide a procedure analogous to the one in Section 2.3 for constructing basic solutions, and prove an analog of Theorem 2.4.

Exercise 2.4 We know that every linear programming problem can be converted to an equivalent problem in standard form. We also know that nonempty polyhedra in standard form have at least one extreme point. We are then tempted to conclude that every nonempty polyhedron has at least one extreme point. Explain what is wrong with this argument.

Exercise 2.5 (Extreme points of isomorphic polyhedra) A mapping f is called *affine* if it is of the form $f(\mathbf{x}) = \mathbf{Ax} + \mathbf{b}$, where $\mathbf{A}$ is a matrix and $\mathbf{b}$ is a vector. Let P and Q be polyhedra in $\Re^n$ and $\Re^m$, respectively. We say that P and Q are *isomorphic* if there exist affine mappings $f : P \mapsto Q$ and $g : Q \mapsto P$ such that $g\big(f(\mathbf{x})\big) = \mathbf{x}$ for all $\mathbf{x} \in P$, and $f\big(g(\mathbf{y})\big) = \mathbf{y}$ for all $\mathbf{y} \in Q$. (Intuitively, isomorphic polyhedra have the same shape.)

(a) If P and Q are isomorphic, show that there exists a one-to-one correspondence between their extreme points. In particular, if f and g are as above, show that $\mathbf{x}$ is an extreme point of P if and only if $f(\mathbf{x})$ is an extreme point of Q.

(b) **(Introducing slack variables leads to an isomorphic polyhedron)** Let $P = \{\mathbf{x} \in \Re^n \mid \mathbf{Ax} \geq \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$, where $\mathbf{A}$ is a matrix of dimensions $k \times n$. Let $Q = \{(\mathbf{x}, \mathbf{z}) \in \Re^{n+k} \mid \mathbf{Ax} - \mathbf{z} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0},\ \mathbf{z} \geq \mathbf{0}\}$. Show that P and Q are isomorphic.

Exercise 2.6 (Carathéodory's theorem) Let $\mathbf{A}_1, \ldots, \mathbf{A}_n$ be a collection of vectors in $\Re^m$.

(a) Let

$$C = \left\{ \sum_{i=1}^{n} \lambda_i \mathbf{A}_i \;\middle|\; \lambda_1, \ldots, \lambda_n \geq 0 \right\}.$$

Show that any element of C can be expressed in the form $\sum_{i=1}^n \lambda_i \mathbf{A}_i$, with $\lambda_i \geq 0$, and with at most m of the coefficients λ_i being nonzero. *Hint:* Consider the polyhedron

$$\Lambda = \left\{ (\lambda_1, \ldots, \lambda_n) \in \Re^n \;\middle|\; \sum_{i=1}^{n} \lambda_i \mathbf{A}_i = \mathbf{y},\ \lambda_1, \ldots, \lambda_n \geq 0 \right\}.$$

(b) Let P be the convex hull of the vectors $\mathbf{A}_i$:

$$P = \left\{ \sum_{i=1}^{n} \lambda_i \mathbf{A}_i \;\middle|\; \sum_{i=1}^{n} \lambda_i = 1,\ \lambda_1, \ldots, \lambda_n \geq 0 \right\}.$$

Show that any element of P can be expressed in the form $\sum_{i=1}^n \lambda_i \mathbf{A}_i$, where $\sum_{i=1}^n \lambda_i = 1$ and $\lambda_i \geq 0$ for all i, with at most $m + 1$ of the coefficients λ_i being nonzero.

Exercise 2.7 Suppose that $\{\mathbf{x} \in \Re^n \mid \mathbf{a}_i'\mathbf{x} \geq b_i,\ i = 1, \ldots, m\}$ and $\{\mathbf{x} \in \Re^n \mid \mathbf{g}_i'\mathbf{x} \geq h_i,\ i = 1, \ldots, k\}$ are two representations of the same nonempty polyhedron. Suppose that the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_m$ span $\Re^n$. Show that the same must be true for the vectors $\mathbf{g}_1, \ldots, \mathbf{g}_k$.

Exercise 2.8 Consider the standard form polyhedron $\{\mathbf{x} \mid \mathbf{Ax} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$, and assume that the rows of the matrix $\mathbf{A}$ are linearly independent. Let $\mathbf{x}$ be a basic solution, and let $J = \{i \mid x_i \neq 0\}$. Show that a basis is associated with the basic solution $\mathbf{x}$ if and only if every column $\mathbf{A}_i$, $i \in J$, is in the basis.

Exercise 2.9 Consider the standard form polyhedron $\{\mathbf{x} \mid \mathbf{Ax} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$, and assume that the rows of the matrix $\mathbf{A}$ are linearly independent.

(a) Suppose that two different bases lead to the same basic solution. Show that the basic solution is degenerate.

(b) Consider a degenerate basic solution. Is it true that it corresponds to two or more distinct bases? Prove or give a counterexample.

(c) Suppose that a basic solution is degenerate. Is it true that there exists an adjacent basic solution which is degenerate? Prove or give a counterexample.

Exercise 2.10 Consider the standard form polyhedron $P = \{\mathbf{x} \mid \mathbf{Ax} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$. Suppose that the matrix $\mathbf{A}$ has dimensions $m \times n$ and that its rows are linearly independent. For each one of the following statements, state whether it is true or false. If true, provide a proof, else, provide a counterexample.

(a) If $n = m + 1$, then P has at most two basic feasible solutions.

(b) The set of all optimal solutions is bounded.

(c) At every optimal solution, no more than m variables can be positive.

(d) If there is more than one optimal solution, then there are uncountably many optimal solutions.

(e) If there are several optimal solutions, then there exist at least two basic feasible solutions that are optimal.

(f) Consider the problem of minimizing $\max\{\mathbf{c}'\mathbf{x}, \mathbf{d}'\mathbf{x}\}$ over the set P. If this problem has an optimal solution, it must have an optimal solution which is an extreme point of P.

Exercise 2.11 Let $P = \{\mathbf{x} \in \Re^n \mid \mathbf{Ax} \geq \mathbf{b}\}$. Suppose that at a particular basic feasible solution, there are k active constraints, with $k > n$. Is it true that there exist exactly $\binom{k}{n}$ bases that lead to this basic feasible solution? Here $\binom{k}{n} = k!/\big(n!(k-n)!\big)$ is the number of ways that we can choose n out of k given items.

Exercise 2.12 Consider a nonempty polyhedron P and suppose that for each variable x_i we have either the constraint $x_i \geq 0$ or the constraint $x_i \leq 0$. Is it true that P has at least one basic feasible solution?

Exercise 2.13 Consider the standard form polyhedron $P = \{\mathbf{x} \mid \mathbf{Ax} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$. Suppose that the matrix $\mathbf{A}$, of dimensions $m \times n$, has linearly independent rows, and that all basic feasible solutions are nondegenerate. Let $\mathbf{x}$ be an element of P that has exactly m positive components.

(a) Show that $\mathbf{x}$ is a basic feasible solution.

(b) Show that the result of part (a) is false if the nondegeneracy assumption is removed.

Exercise 2.14 Let P be a bounded polyhedron in $\Re^n$, let $\mathbf{a}$ be a vector in $\Re^n$, and let b be some scalar. We define

$$Q = \{\mathbf{x} \in P \mid \mathbf{a}'\mathbf{x} = b\}.$$

Show that every extreme point of Q is either an extreme point of P or a convex combination of two adjacent extreme points of P.

Exercise 2.15 (Edges joining adjacent vertices) Consider the polyhedron $P = \{\mathbf{x} \in \Re^n \mid \mathbf{a}_i'\mathbf{x} \geq b_i,\ i = 1, \ldots, m\}$. Suppose that $\mathbf{u}$ and $\mathbf{v}$ are distinct basic feasible solutions that satisfy $\mathbf{a}_i'\mathbf{u} = \mathbf{a}_i'\mathbf{v} = b_i$, $i = 1, \ldots, n-1$, and that the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}$ are linearly independent. (In particular, $\mathbf{u}$ and $\mathbf{v}$ are adjacent.) Let $L = \{\lambda\mathbf{u} + (1-\lambda)\mathbf{v} \mid 0 \leq \lambda \leq 1\}$ be the segment that joins $\mathbf{u}$ and $\mathbf{v}$. Prove that $L = \{\mathbf{z} \in P \mid \mathbf{a}_i'\mathbf{z} = b_i,\ i = 1, \ldots, n-1\}$.

Exercise 2.16 Consider the set $\{\mathbf{x} \in \Re^n \mid x_1 = \cdots = x_{n-1} = 0,\ 0 \leq x_n \leq 1\}$. Could this be the feasible set of a problem in standard form?

Exercise 2.17 Consider the polyhedron $\{\mathbf{x} \in \Re^n \mid \mathbf{Ax} \leq \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$ and a nondegenerate basic feasible solution $\mathbf{x}^*$. We introduce slack variables $\mathbf{z}$ and construct a corresponding polyhedron $\{(\mathbf{x}, \mathbf{z}) \mid \mathbf{Ax} + \mathbf{z} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0},\ \mathbf{z} \geq \mathbf{0}\}$ in standard form. Show that $(\mathbf{x}^*, \mathbf{b} - \mathbf{Ax}^*)$ is a nondegenerate basic feasible solution for the new polyhedron.

Exercise 2.18 Consider a polyhedron $P = \{\mathbf{x} \mid \mathbf{Ax} \geq \mathbf{b}\}$. Given any $\epsilon > 0$, show that there exists some $\overline{\mathbf{b}}$ with the following two properties:

(a) The absolute value of every component of $\mathbf{b} - \overline{\mathbf{b}}$ is bounded by ϵ.

(b) Every basic feasible solution in the polyhedron $P = \{\mathbf{x} \mid \mathbf{Ax} \geq \overline{\mathbf{b}}\}$ is nondegenerate.

Exercise 2.19* Let $P \subset \Re^n$ be a polyhedron in standard form whose definition involves m linearly independent equality constraints. Its dimension is defined as the smallest integer k such that P is contained in some k-dimensional affine subspace of $\Re^n$.

(a) Explain why the dimension of P is at most $n - m$.

(b) Suppose that P has a nondegenerate basic feasible solution. Show that the dimension of P is equal to $n - m$.

(c) Suppose that $\mathbf{x}$ is a degenerate basic feasible solution. Show that $\mathbf{x}$ is degenerate under every standard form representation of the same polyhedron (in the same space $\Re^n$). *Hint:* Using parts (a) and (b), compare the number of equality constraints in two representations of P under which $\mathbf{x}$ is degenerate and nondegenerate, respectively. Then, count active constraints.

Exercise 2.20* Consider the Fourier-Motzkin elimination algorithm.

(a) Suppose that the number m of constraints defining a polyhedron P is even. Show, by means of an example, that the elimination algorithm may produce a description of the polyhedron $\Pi_{n-1}(P)$ involving as many as $m^2/4$ linear constraints, but no more than that.

(b) Show that the elimination algorithm produces a description of the one-dimensional polyhedron $\Pi_1(P)$ involving no more than $m^{2^{n-1}}/2^{2^n-2}$ constraints.

(c) Let $n = 2^p + p + 2$, where p is a nonnegative integer. Consider a polyhedron in $\Re^n$ defined by the $8\binom{n}{3}$ constraints

$$\pm x_i \pm x_j \pm x_k \leq 1, \qquad 1 \leq i < j < k \leq n,$$

where all possible combinations are present. Show that after p eliminations, we have at least

$$2^{2^p+2}$$

constraints. (Note that this number increases exponentially with n.)

Exercise 2.21 Suppose that Fourier-Motzkin elimination is used in the manner described at the end of Section 2.8 to find the optimal cost in a linear programming problem. Show how this approach can be augmented to obtain an optimal solution as well.

Exercise 2.22 Let P and Q be polyhedra in $\Re^n$. Let $P + Q = \{\mathbf{x} + \mathbf{y} \mid \mathbf{x} \in P,\ \mathbf{y} \in Q\}$.

(a) Show that $P + Q$ is a polyhedron.

(b) Show that every extreme point of $P + Q$ is the sum of an extreme point of P and an extreme point of Q.

2.11 Notes and sources

The relation between algebra and geometry goes far back in the history of mathematics, but was limited to two and three-dimensional spaces. The insight that the same relation goes through in higher dimensions only came in the middle of the nineteenth century.

2.2. Our algebraic definition of basic (feasible) solutions for general polyhedra, in terms of the number of linearly independent active constraints, is not common. Nevertheless, we consider it to be quite central, because it provides the main bridge between the algebraic and geometric viewpoint, it allows for a unified treatment, and shows that there is not much that is special about standard form problems.

2.8. Fourier-Motzkin elimination is due to Fourier (1827), Dines (1918), and Motzkin (1936).