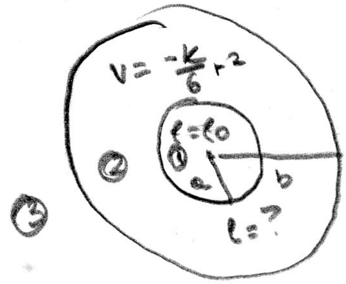
Pauta Ejercicio 2 Francisco Parra O



se pide $\vec{E}, V, \sigma_1, \sigma_2, \rho$

En ①

$$\oint \vec{E}\cdot d\vec{s} = \frac{Q}{\varepsilon_0} \Rightarrow 4\pi r^2 E_r = \frac{Q}{\varepsilon_0}$$

Pero $Q = \int_0^r \int_0^\pi \int_0^{2\pi} \rho_0 r'^2 \operatorname{sen}\theta \, d\theta \, d\phi \, dr$

$$= 4\pi \rho_0 \frac{r'^3}{3}\Big|_0^r = \frac{\cancel{4\pi}\rho_0 \cdot r^3}{3} = E_r \cancel{4\pi} r^2$$

$$\Rightarrow \boxed{E_r = \frac{\rho_0 r}{3}}$$

En ②

$$\vec{E} = -\nabla V = -\nabla\left(-\frac{k}{6}r^2\right) = \frac{\partial}{\partial r}\left(\frac{k}{6}r^2\right)\hat{r} = \boxed{\frac{kr}{3}\hat{r}}$$

Ahora, para sacar la densidad, usamos Gauss, pero de forma inversa

$$\oint \vec{E}\cdot d\vec{s} = \frac{Q}{\varepsilon_0} \Rightarrow E_r \cdot 4\pi r^2 = \frac{Q}{\varepsilon_0}$$

Pero $Q = \int_{V_1} \rho_1 + \int_S \sigma_1 + \int_{V_2} \rho_2$

$$\int_{V_1} \rho_1 = \int_0^a \rho_0 r^2 \operatorname{sen}\theta \, dr \, d\theta \, d\phi = \rho_0 \frac{a^3}{3} \cdot 4\pi$$

$$\int_S \sigma_1 = \sigma_1 \cdot r^2 \cdot 4\pi \Big|_a$$

$$\int_{V_2} \rho_2 = \int_a^r \int_0^\pi \int_0^{2\pi} \rho \cdot r'^2 \operatorname{sen}\theta \, d\phi = 4\pi \int_a^r \rho(r') r'^2 \, dr'$$

$$\Rightarrow E_r 4\pi r^2 \varepsilon_0 = \rho_0 \frac{a^3}{3} \cdot 4\pi + \sigma_1 r^2 \cdot 4\pi + 4\pi \int_a^r \rho(r') r'^2 \, dr'$$

$$4\pi r^2 \varepsilon_0 \cdot \frac{kr}{3} = \rho_0 \frac{a^3}{3} \cdot 4\pi + \sigma_1 a^2 \cdot 4\pi + 4\pi \int_a^r r'^2 \rho(r') dr'$$

Derivando con respecto a r

$$\cancel{4\pi} r^2 \varepsilon_0 \cdot k = 0 + 0 + \cancel{4\pi} r^2 \rho(r)$$

$$r \varepsilon_0 k = 0 + \rho(r) r$$

$$\Rightarrow \boxed{\varepsilon_0 k = \rho(r)}$$

Finalmente, para el campo eléctrico externo

$$\oint \vec{E} \cdot d\vec{s} = \frac{Q}{\varepsilon_0}$$, Donde Q está determinado por

$$Q = \int_{V_1} \rho_1 + \int_{S_1} \sigma_1 + \int_{V_2} \rho_2 + \int_{S_2} \sigma_2$$

$$\int_{V_1} \rho_1 = \frac{\rho_0 a^3}{3} \cdot 4\pi \qquad \int_{S_1} \sigma_1 = \sigma_1 a^2 \cdot 4\pi$$

$$\int_{V_2} \rho_2 = \int_0^{\pi} \int_0^{2\pi} \int_a^b \varepsilon_0 k \, r^2 \operatorname{sen}\theta \, d\theta \, d\phi = 4\pi \varepsilon_0 k \frac{r^3}{3} \Big|_a^b = \frac{4\pi \varepsilon_0 k}{3} (b^3 - a^3)$$

$$\int_S \sigma_2 = \sigma_2 \cdot b^2 \cdot 4\pi$$

$$\Rightarrow \cancel{4\pi} r^2 E_r \varepsilon_0 = \cancel{4\pi} \left(\frac{\rho_0 a^3}{3} + \frac{\varepsilon_0 k b^3}{3} - \frac{\varepsilon_0 k a^3}{3} + \sigma_1 a^2 + \sigma_2 b^2 \right)$$

$$E_r \varepsilon_0 = \frac{1}{r^2} \underbrace{\left(\frac{\rho_0 a^3}{3} + \frac{\varepsilon_0 k b^3}{3} - \frac{\varepsilon_0 k a^3}{3} + \sigma_1 a^2 + \sigma_2 b^2 \right)}_{A}$$

ahora, el potencial en el exterior

$$\Delta V = -\int \vec{E}\cdot d\vec{r}$$

$$V(\cancel{\infty})^{0} - V(r) = -\int_r^{\infty} \vec{E}\cdot\hat{r}\,dr'$$

$$-V(r) = -\int_r^{\infty} \frac{A}{r'^2}\,dr' \Rightarrow V(r) = A\cdot\left(-\frac{1}{r'}\right)_r^{\infty}$$

$$= -A\left(\frac{1}{r'}\right)_r^{\infty} = -A\left(0-\frac{1}{r}\right)$$

$$= \frac{A}{r\varepsilon_0}$$

$$\Rightarrow \boxed{V_{③}(r) = \frac{A}{r\varepsilon_0} = \frac{1}{r\varepsilon_0}\left(\frac{\rho_0 a^3}{3} + \frac{\varepsilon_0 k b^3}{3} - \frac{\varepsilon_0 k a^3}{3} + \sigma_1 a^2 + \sigma_2 b^2\right)}$$

y el Potencial en la zona 1

$$V(a) - V(r) = -\int_r^{a} \vec{E}\cdot d\vec{r}\,, \quad \text{pero } V(a) = -\frac{k}{6}a^2$$

$$-\frac{k}{6}a^2 - V(r) = -\int_r^{a} \frac{\rho_0 r'}{3}\,dr' = -\frac{\rho_0 r'^2}{6}\Big|_r^{a} = -\frac{\rho_0}{6}\left(a^2 - r^2\right)$$

$$= -\frac{\rho_0 a^2}{6} + \frac{\rho_0 r^2}{6}$$

$$\Rightarrow \boxed{V_{①}(r) = \frac{1}{6}\left(\rho_0 a^2 - k a^2 - \rho_0 r^2\right)}$$

Y con las condiciones de borde determinamos σ_1 y σ_2

$$V_{②}(b) = V_{③}(b) \Rightarrow -\frac{k}{6}b^2 = \frac{1}{\varepsilon_0 b}\left(\frac{\rho_0 a^3}{3} + \frac{\varepsilon_0 k b^3}{3} - \frac{\varepsilon_0 k a^3}{3} + \sigma_1 a^2 + \sigma_2 b^2\right)$$

Y además

$$E_r\cdot \cancel{4\pi} r^2 \varepsilon_0 = \frac{\rho_0 a^3}{3}\cdot\cancel{4\pi} + \sigma_1 a^2\cdot\cancel{4\pi} + \cancel{4\pi}\frac{\varepsilon_0 k}{3}(r^3 - a^3)$$

$$E_r\cdot r^2\varepsilon_0 = \frac{\rho_0 a^3}{3} + \sigma_1 a^2 + \frac{\varepsilon_0 k r^3}{3} - \frac{\varepsilon_0 k a^3}{3}$$

De aquí inferimos que (por la forma del campo)

$$\frac{\rho_0 a^3}{3} + \sigma_1 a^2 - \frac{\varepsilon_0 k a^3}{3} = 0$$

$$\Rightarrow \sigma_1 a^{\cancel{2}} = \frac{a^{\cancel{3}}}{3}(\varepsilon_0 k - \rho_0)$$

$$\boxed{\sigma_1 = \frac{a}{3}(\varepsilon_0 k - \rho_0)}$$

Con esto, calculamos σ_2

$$-\frac{kb^3}{6} = \frac{1}{\varepsilon_0}\left(\frac{\rho_0 a^3}{3} + \frac{\varepsilon_0 k b^3}{3} - \frac{\varepsilon_0 k a^3}{3} + \sigma_1 a^2 + \sigma_2 b^2\right)$$

$$\Rightarrow -\frac{kb^3}{6} = \frac{1}{\varepsilon_0}\left(\cancel{\frac{\rho_0 a^3}{3}} + \frac{\varepsilon_0 k b^3}{3} - \cancel{\frac{\varepsilon_0 k a^3}{3}} + \cancel{\frac{a^3}{3}\varepsilon_0 k} - \cancel{\frac{a^3}{3}\rho_0} + \sigma_2 b^2\right)$$

$$-\frac{\varepsilon_0 k b^{\cancel{3}}}{6} = \frac{\varepsilon_0 k b^{\cancel{3}}}{3} + \sigma_2 \cancel{b^2}$$

$$\sigma_2 = -\varepsilon_0 k b\left(\frac{1}{6} + \frac{1}{3}\right)$$

$$\boxed{\sigma_2 = -\frac{\varepsilon_0 k b}{2}}$$