Minimal surfaces and entire solutions of the Allen-Cahn equation

Manuel del Pino

DIM and CMM

Universidad de Chile

U. de Chile MA 725 Noviembre 2010

The Allen-Cahn Equation

$$\text{(AC)} \qquad \Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^n$$

Euler-Lagrange equation for the *energy functional*

$$J(u) = \frac{1}{2}\int |\nabla u|^2 + \frac{1}{4}\int (1-u^2)^2$$

$u = +1$ and $u = -1$ are *global minimizers* of the energy representing, in the gradient theory of phase transitions, two distinct phases of a material.

Of interest are solutions of (AC) that connect these two values. They represent states in which the two phases coexist.

The case $N = 1$. The function

$$w(t) := \tanh\left(\frac{t}{\sqrt{2}}\right)$$

connects monotonically -1 and $+1$ and solves

$$w'' + w - w^3 = 0, \quad w(\pm\infty) = \pm 1, \quad w' > 0.$$

For any $p, \nu \in \mathbb{R}^N$, $|\nu| = 1$, the functions

$$u(x) := w(z), \quad z = (x - p) \cdot \nu$$

solve equation (AC). $z =$ normal coordinate to the hyperplane through p, unit normal ν.

De Giorgi's conjecture (1978): *Let* u *be a bounded solution of equation*

$$\text{(AC)} \qquad \Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^N,$$

which is monotone in one direction, say $\partial_{x_N} u > 0$. *Then,* **at least** *when* $N \leq 8$, *there exist* p, ν *such that*

$$u(x) = w(\,(x - p) \cdot \nu).$$

This statement is equivalent to:

At least when $N \leq 8$, all level sets of u, $[u = \lambda]$ must be hyperplanes.

Parallel to **Bernstein's conjecture** for minimal surfaces which are entire graphs.

Entire minimal graph in $\mathbb{R}^N$:

$$\Gamma = \{(x', F(x')) \in \mathbb{R}^{N-1} \times \mathbb{R} \; / \; x' \in \mathbb{R}^{N-1}\}$$

where F solves the minimal surface equation

$$H_\Gamma \; := \; \nabla \cdot \left(\frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = 0 \quad \text{in } \mathbb{R}^{N-1}. \qquad (MS)$$

Bernstein's conjecture: *All entire minimal graphs are hyperplanes, namely any entire solution of (MS) must be a linear affine function*:

True for $N \leq 8$:
- Bernstein (1910), Fleming (1962) $N = 3$
- De Giorgi (1965) $N = 4$
- Almgren (1966), $N = 5$
- Simons (1968), $N = 6, 7, 8$.

False for $N \geq 9$: Bombieri-De Giorgi-Giusti found a counterexample (1969).

De Giorgi's Conjecture: *u bounded solution of (AC), $\partial_{x_N} u > 0$ then level sets $[u = \lambda]$ are hyperplanes.*

- True for $N = 2$. Ghoussoub and Gui (1998).
- True for $N = 3$. Ambrosio and Cabré (1999).
- True for $4 \leq N \leq 8$ (Savin (2009), thesis (2003)) if in addition

$$\lim_{x_N \to \pm\infty} u(x', x_N) = \pm 1 \quad \text{for all} \quad x' \in \mathbb{R}^{N-1}.$$

The Bombieri-De Giorgi-Giusti minimal graph:

Explicit construction by super and sub-solutions. $N = 9$:

$$H(F) := \nabla \cdot \left(\frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = 0 \quad \text{in } \mathbb{R}^8.$$

$$F : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}, \quad (\mathbf{u}, \mathbf{v}) \mapsto F(|\mathbf{u}|, |\mathbf{v}|).$$

In addition, $F(|\mathbf{u}|, |\mathbf{v}|) > 0$ for $|\mathbf{v}| > |\mathbf{u}|$ and

$$F(|\mathbf{u}|, |\mathbf{v}|) = -F(|\mathbf{v}|, |\mathbf{u}|).$$

Polar coordinates:

$$|\mathbf{u}| = r\cos\theta,\ \ |\mathbf{v}| = r\sin\theta, \quad \theta \in (0, \frac{\pi}{2})$$

Mean curvature operator at $F = F(r, \theta)$

$$H[F] \ = \ \frac{1}{r^7 \sin^3 2\theta}\partial_r \left(\frac{F_r r^7 \sin^3 2\theta}{\sqrt{1 + F_r^2 + r^{-2}F_\theta^2}} \right) + \frac{1}{r^7 \sin^3 2\theta}\partial_\theta \left(\frac{F_\theta r^5 \sin^3 2\theta}{\sqrt{1 + F_r^2 + r^{-2}F_\theta^2}} \right).$$

Separation of variables $F_0(r, \theta) = r^3 g(\theta)$.

$$H[F_0] \;=\; \frac{1}{r^7 \sin^3 2\theta}\partial_r\left(\frac{3r^7 g\,\sin^3 2\theta}{\sqrt{r^{-4}+9g^2+g'^2}}\right) + \frac{1}{r\sin^3 2\theta}\partial_\theta\left(\frac{g'\sin^3 2\theta}{\sqrt{r^{-4}+9g^2+g'^2}}\right) .$$

As $r \to \infty$ the equation $H(F_0) = 0$ becomes the ODE

$$\frac{21g\,\sin^3 2\theta}{\sqrt{9g^2+g'^2}} + \left(\frac{g'\sin^3 2\theta}{\sqrt{9g^2+g'^2}}\right)' = 0 \quad \text{in } \left(\frac{\pi}{4},\frac{\pi}{2}\right),$$

$$g\left(\frac{\pi}{4}\right) = 0 = g'\left(\frac{\pi}{2}\right).$$

This problem has a solution g positive in $(\frac{\pi}{4},\frac{\pi}{2})$.

We check directly that

• $F_0(r,\theta) = r^3 g(\theta)$ is a subsolution of the minimal surface equation $H(F) = 0$: $H(F_0) \geq 0$

• $F_0(r,\theta)$ accurate approximation to a solution of the minimal surface equation:

$$H(F_0) = O(r^{-5}) \quad \text{as } r \to +\infty.$$

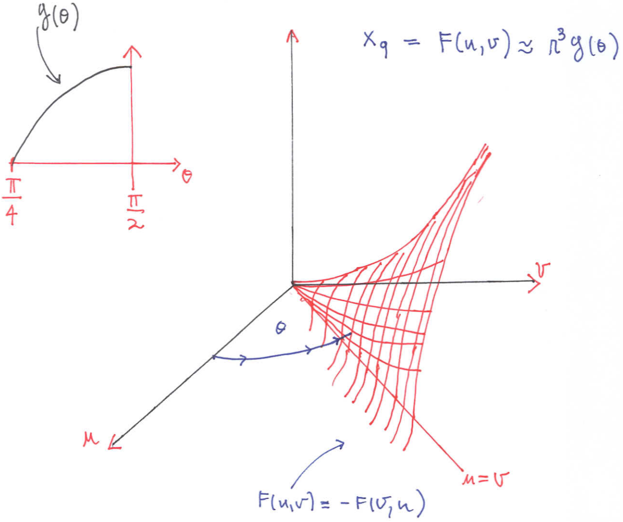
The supersolution of Bombieri, De Giorgi and Giusti can be refined to yield that F_0 gives the precise asymptotic behavior of F.

Refinement of asymptotic behavior of BDG surface $x_9 = F(r, \theta)$, (D., Kowalczyk, Wei (2008)):

For $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$ *we have, for* $0 < \sigma < 1$ *and all large* r,

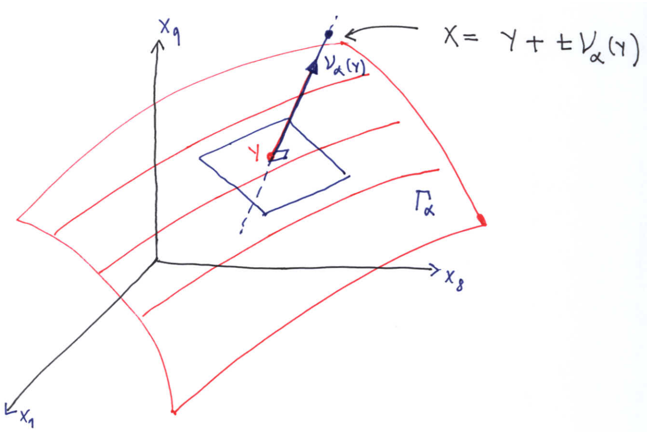
$$\boxed{F_0(r,\theta) \ \le \ F(r,\theta) \le F_0(r,\theta) + Ar^{-\sigma} \quad \text{as } r \to +\infty.}$$

The BDG surface:



Let $\nu_\varepsilon(y) := \nu(\varepsilon y), \quad y \in \Gamma_\varepsilon = \varepsilon^{-1}\Gamma$ be unit normal with $\nu_9 > 0$.
Local coordinates in in a tubular neighborhood of Γ_ε:

$$\boxed{x = y + \zeta \nu_\varepsilon(y), \quad y \in \Gamma_\varepsilon, \quad |\zeta| < \frac{\delta}{\varepsilon}}$$



Theorem (D., Kowalczyk, Wei (2008))

Let Γ *be a BDG minimal graph in* $\mathbb{R}^9$ *and* $\Gamma_\varepsilon := \varepsilon^{-1}\Gamma$. *Then for all small* $\varepsilon > 0$, *there exists a bounded solution* u_ε *of* (AC), *monotone in the* x_9*-direction, with*

$$u_\varepsilon(x) = w(\zeta) + O(\varepsilon), \quad x = y + \zeta\nu(\varepsilon y), \quad y \in \Gamma_\varepsilon, \ |\zeta| < \frac{\delta}{\varepsilon},$$

$$\lim_{x_9 \to \pm\infty} u(x', x_9) = \pm 1 \quad \textit{for all} \quad x' \in \mathbb{R}^8.$$

u_ε is a "counterexample" to De Giorgi's conjecture in dimension 9 (hence in any dimension higher).

Sketch of the proof

Let Γ be a fixed BDG graph and let ν designate a choice of its unit normal. Local coordinates near Γ:

$$x = y + z\nu(y), \quad y \in \Gamma, \ |z| < \delta$$

Laplacian in these coordinates:

$$\boxed{\Delta_x \,=\, \partial_{zz} \,+\, \Delta_{\Gamma^z} \,-\, H_{\Gamma_z}(y)\,\partial_z}$$

$$\Gamma^z := \{y + z\nu(y) \ / \ y \in \Gamma\}.$$

Δ_{Γ^z} is the Laplace-Beltrami operator on Γ^z acting on functions of y, and $H_{\Gamma^z}(y)$ its mean curvature at the point $y + z\nu(y)$.

Let $k_1, \dots, k_N$ denote the principal curvatures of Γ. Then

$$\boxed{H_{\Gamma^z} = \sum_{i=1}^{8} \frac{k_i}{1 - z k_i}}$$

For later reference, we expand

$$H_{\Gamma^z}(y) = H_\Gamma(y) + z\,|A_\Gamma(y)|^2 + z^2 \sum_{i=1}^{N} k_i^3 + \cdots$$

where

$$\underbrace{H_\Gamma = \sum_{i=1}^{8} k_i\,,}_{mean\ \ curvature} \qquad \underbrace{|A_\Gamma|^2 = \sum_{i=1}^{8} k_i^2}_{norm\ \ second\ \ fundamental\ \ form} \qquad .$$

Letting $f(u) = u - u^3$ the equation

$$\Delta u + f(u) = 0 \quad \text{in } \mathbb{R}^9$$

becomes, for

$$u(y, \zeta) := u(x), \quad x = y + \zeta\nu(\varepsilon y), \quad y \in \Gamma_\varepsilon, \ |\zeta| < \delta/\varepsilon,$$

ν unit normal to Γ with $\nu_N > 0$,

$$S(u) := \Delta u + f(u) \ =$$

$$\Delta_{\Gamma_\varepsilon^\zeta} u - \varepsilon H_{\Gamma^{\varepsilon\zeta}}(\varepsilon y)\,\partial_\zeta u \ + \partial_\zeta^2 u + f(u) \ = \ 0.$$

- We look for a solution of the form (near Γ_ε)

$$u_\varepsilon(x) = w(\zeta - \varepsilon h(\varepsilon y)) + \phi, \quad x = y + \zeta\nu(\varepsilon y)$$

 for a function h defined on Γ, left as a parameter to be adjusted and ϕ small.
- Let $r(y', y_9) = |y'|$. We assume a priori on h that

$$\|(1+r^3)D_\Gamma^2 h\|_{L^\infty(\Gamma)} + \|(1+r^2)D_\Gamma h\|_{L^\infty(\Gamma)} + \|(1+r)h\|_{L^\infty(\Gamma)} \le M$$

 for some large, fixed number M.

Let us change variables to $t = \zeta - \varepsilon h(\varepsilon y)$, or

$$u(y,t) := u(x) \quad x = y + (t + \varepsilon h(\varepsilon y))\, \nu(\varepsilon y)$$

The equation becomes

$$S(u) = \partial_{tt} u + \Delta_{\Gamma_\varepsilon^\zeta} u - \varepsilon H_{\Gamma^{\varepsilon\zeta}}(\varepsilon y)\, \partial_t u +$$

$$+ \varepsilon^4 |\nabla_{\Gamma^{\varepsilon\zeta}} h(\varepsilon y)|^2 \partial_{tt} u - 2\varepsilon^3 \langle \nabla_{\Gamma^{\varepsilon\zeta}} h(\varepsilon y), \partial_t \nabla_{\Gamma^{\varepsilon\zeta}} u \rangle$$
$$- \varepsilon^3 \Delta_{\Gamma^{\varepsilon\zeta}} h(\varepsilon y)\, \partial_t u + f(u) \; = \; 0, \quad \zeta = t + \varepsilon h(\varepsilon y).$$

Look for solution u_ε of the form

$$u_\varepsilon(t,y) = w(t) + \phi(t,y)$$

for a small function ϕ.

$$u_\varepsilon(t,y) = w(t) + \phi(t,y)$$

The equation in terms of ϕ becomes

$$\partial_{tt}\phi + \Delta_{\Gamma_\varepsilon}\phi + B\phi + f'(w(t))\phi + N(\phi) + E \; = \; 0.$$

where B is a small linear second order operator, and

$$E = S(w(t)), \quad N(\phi) = f(w+\phi) - f(w) - f'(w)\phi \approx f''(w)\phi^2.$$

The error of approximation.

$$E := S(w(t)) \ =$$

$$\varepsilon^4 |\nabla_{\Gamma^{\varepsilon\zeta}} h(\varepsilon y)|^2 w''(t) - [\varepsilon^3 \Delta_{\Gamma^{\varepsilon\zeta}} h(\varepsilon y) \ + \varepsilon H_{\Gamma^{\varepsilon\zeta}}(\varepsilon y)]\, w'(t),$$

and

$$\varepsilon H_{\Gamma^{\varepsilon\zeta}}(\varepsilon y) = \varepsilon^2 (t+\varepsilon h(\varepsilon y)) |A_\Gamma(\varepsilon y)|^2 + \varepsilon^3 (t+\varepsilon h(\varepsilon y))^2 \sum_{i=1}^{8} k_i^3(\varepsilon y) + \cdots$$

A crucial fact: (L. Simon (1989)) $k_i = O(r^{-1})$ as $r \to +\infty$. In particular

$$|E(y,t)| \ \le \ C\varepsilon^2 r(\varepsilon y)^{-2}.$$

Equation

$$\partial_{tt}\phi + \Delta_{\Gamma_\varepsilon}\phi + B\phi + f'(w(t))\phi + N(\phi) + E \;=\; 0.$$

makes sense only for $|t| < \delta\varepsilon^{-1}$.

A **gluing procedure** reduces the full problem to

$$\boxed{\partial_{tt}\phi + \Delta_{\Gamma_\varepsilon}\phi + B\phi + f'(w)\phi + N(\phi) + E = 0 \quad \text{in } \mathbb{R}\times\Gamma_\varepsilon,}$$

where E and B are the same as before, but cut-off far away. N is modified by the addition of a small nonlocal operator of ϕ.

We find a small solution to this problem in **two steps**.

Infinite dimensional Lyapunov-Schmidt reduction:

Step 1: Given the parameter function h, find a a solution $\phi = \Phi(h)$ to the problem

$$\partial_{tt}\phi + \Delta_{\Gamma_\varepsilon}\phi + B\phi + f'(w(t))\phi + N(\phi) + E = c(y)w'(t) \quad \text{in } \mathbb{R}\times\Gamma_\varepsilon,$$

$$\int_{\mathbb{R}} \phi(t,y)w'(t)\,dt = 0 \quad \text{for all} \quad y \in \Gamma_\varepsilon.$$

Step 2: Find a function h such that for all $y \in \Gamma_\varepsilon$,

$$c(y) := \frac{1}{\int_{\mathbb{R}} w'^2\,dt}\int_{\mathbb{R}} (E + B\Phi(h) + N(\Phi(h)))\,w'\,dt = 0.$$

For **Step 1** we solve first the linear problem

$$\partial_{tt}\phi + \Delta_{\Gamma_\varepsilon}\phi + f'(w(t))\phi = g(t,y) - c(y)w'(t) \quad \text{in } \mathbb{R}\times\Gamma_\varepsilon$$

$$\int_{\mathbb{R}} \phi(y,t)w'(t)\,dt = 0 \quad \text{in } \Gamma_\varepsilon, \ \ c(y) := \frac{\int_{\mathbb{R}} g(y,t)w'(t)\,dt}{\int_{\mathbb{R}} w'^2\,dt}.$$

There is a unique bounded solution $\phi := A(g)$ *if* g *is bounded. Moreover, for any* $\nu \ge 0$ *we have*

$$\|(1+r(\varepsilon y)^\nu)\phi\|_\infty \ \le \ C\,\|(1+r(\varepsilon y))^\nu g\|_\infty.$$

$\Gamma_\varepsilon \approx \mathbb{R}^{N-1}$ around each of its points as $\varepsilon \to 0$, in uniform way. The problem is qualitatively similar to Γ_ε replaced with $\mathbb{R}^{N-1}$.

Fact: *The linear model problem*

$$\partial_{tt}\phi + \Delta_y \phi + f'(w(t))\phi = g(t,y) - c(y)w'(t) \quad \textit{in } \mathbb{R}^N$$

$$\int_{\mathbb{R}} \phi(y,t)w'(t)\,dt = 0 \quad \textit{in } \mathbb{R}^{N-1}, \quad c(y) := \frac{\int_{\mathbb{R}} g(y,t)w'(t)\,dt}{\int_{\mathbb{R}} w'^2\,dt}$$

has a unique bounded solution ϕ *if* g *is bounded, and*

$$\|\phi\|_\infty \ \le \ C\,\|g\|_\infty.$$

Let us prove first the a priori estimate:

If the a priori estimate did not hold, there would exist

$$\|\phi_n\|_\infty = 1, \quad \|g_n\|_\infty \to 0,$$

$$\partial_{tt}\phi_n + \Delta_y \phi_n + f'(w(t))\phi_n = g_n(t,y), \quad \int_{\mathbb{R}} \phi_n(y,t) w'(t)\, dt = 0.$$

Using maximum principle and local elliptic estimates, we may assume that $\phi_n \to \phi \neq 0$ uniformly over compact sets where

$$\partial_{tt}\phi + \Delta_y \phi + f'(w(t))\phi = 0, \quad \int_{\mathbb{R}} \phi(y,t) w'(t)\, dt = 0.$$

Claim: $\phi = 0$, which is a contradiction

A key one-dimensional fact: The spectral gap estimate.

$$L_0(p) := p'' + f'(w(t))p$$

There is a $\gamma > 0$ such that if $p \in H^1(\mathbb{R})$ and $\int_{\mathbb{R}} p\, w'\, dt = 0$ then

$$-\int_{\mathbb{R}} L_0(p)\, p\, dt = \int_{\mathbb{R}} (|p'|^2 - f'(w)p^2)\, dt \ \geq \ \gamma \int_{\mathbb{R}} p^2\, dt\,.$$

Using maximum principle we find $|\phi(y,t)| \le Ce^{-|t|}$. Set $\varphi(y) = \int_{\mathbb{R}} \phi^2(y,t)\,dt$. Then

$$\Delta_y \varphi(y) = 2\int_{\mathbb{R}} \phi\Delta\phi(y,t)\,dt + 2\int_{\mathbb{R}} |\nabla_y \phi(y,t)|^2\,dt \ge$$

$$-2\int_{\mathbb{R}} \phi\partial_{tt}\phi + f'(w)\phi^2\,dt \;=$$

$$2\int_{\mathbb{R}} (|\phi_t|^2 - f'(w)\phi^2)\,dt \ge \gamma\varphi(y).$$

$$-\Delta_y \varphi(y) + \gamma\varphi(y) \le 0$$

and $\varphi \ge 0$ bounded, implies $\varphi \equiv 0$, hence $\phi = 0$, a contradiction.
This proves the a priori estimate.

Existence: take g compactly supported. Set H be the space of all $\phi \in H^1(\mathbb{R}^N)$ with

$$\int_{\mathbb{R}} \phi(y,t) w'(t)\, dt = 0 \quad \text{for all} \quad y \in \mathbb{R}^{N-1}.$$

H is a closed subspace of $H^1(\mathbb{R}^N)$.

The problem: $\phi \in H$ and

$$\partial_{tt}\phi + \Delta_y \phi + f'(w(t))\phi = g(t,y) - w'(t)\frac{\int_{\mathbb{R}} g(y,\tau)w'(\tau)\,d\tau}{\int_{\mathbb{R}} w'^2\,d\tau},$$

can be written variationally as that of minimizing in H the energy

$$I(\phi) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla_y \phi|^2 + |\phi_t|^2 - f'(w)\phi^2 + \int_{\mathbb{R}^N} g\phi$$

I is coercive in H thanks to the 1d spectral gap estimate.
Existence in the general case follows by the L^∞-a priori estimate and approximations.

We write the problem of **Step 1**,

$$\begin{aligned} \partial_{tt}\phi + \Delta_{\Gamma_\varepsilon}\phi + B\phi + f'(w(t))\phi + N(\phi) + E \; &= \\ c(y)w'(t) \quad \text{in } \mathbb{R}\times\Gamma_\varepsilon, \\ \int_{\mathbb{R}} \phi(t,y)w'(t)\,dt \; &= \; 0 \quad \text{for all} \quad y\in\Gamma_\varepsilon, \end{aligned}$$

in fixed point form

$$\phi \; = \; A(B\phi + N(\phi) + E).$$

Contraction mapping principle implies the existence of a unique solution $\phi := \Phi(h)$ with

$$\|(1+r^2(\varepsilon y))\phi\|_\infty \; = \; O(\varepsilon^2).$$

Finally, we carry out **Step 2**. We need to find h such that

$$\int_{\mathbb{R}} [E + B\Phi(h) + N(\Phi(h))] \, (\varepsilon^{-1} y, t) \, w'(t) \, dt = 0 \ \forall \, y \in \Gamma.$$

Since

$$-E(\varepsilon^{-1} y, t) = \varepsilon^2 t w'(t) \, |A_\Gamma(y)|^2 + \varepsilon^3 [\Delta_\Gamma h(y) + |A_\Gamma(y)|^2 h(y) \,] \, w'(t)$$

$$+ \, \varepsilon^3 t^2 w'(t) \, \sum_{j=1}^{8} k_j(y)^3 + \ \textit{smaller terms}$$

the problem becomes

$$\mathcal{J}_\Gamma(h) := \Delta_\Gamma h + |A_\Gamma|^2 h \, = \, c \sum_{i=1}^{8} k_i^3 + \mathcal{N}(h) \quad \text{in } \Gamma,$$

where $\mathcal{N}(h)$ is a small operator.

Fact: *Let* $0 < \sigma < 1$. *Then if*

$$\|(1 + r^{4+\sigma})\, g\|_{L^\infty(\Gamma)} \; < \; +\infty$$

there is a unique solution $h = T(\mathrm{g})$ *to the problem*

$$\mathcal{J}_\Gamma[h] := \Delta_\Gamma h + |A_\Gamma(y)|^2 h = \mathrm{g}(y) \quad \text{in } \Gamma.$$

with

$$\|(1 + r)^{2+\sigma}\, h\|_{L^\infty(\Gamma)} \; \le \; C\, \|(1 + r)^{4+\sigma}\, \mathrm{g}\|_{L^\infty(\Gamma)}\, .$$

We want to solve

$$\mathcal{J}_\Gamma(h) := \Delta_\Gamma h + |A_\Gamma|^2 h \,=\, c\sum_{i=1}^{8} k_i^3 + \mathcal{N}(h) \quad \text{in } \Gamma,$$

using a fixed point formulation for the operator T above.
In $\mathcal{N}(h)$ everything decays $O(r^{-4-\sigma})$, but we only have

$$\sum_{i=1}^{8} k_i^3 = O(r^{-3}).$$

Two more facts:

- There is a function p smooth, with $p(\frac{\pi}{2} - \theta) = -p(\theta)$ for all $\theta \in (0, \frac{\pi}{4})$ such that

$$\sum_{i=1}^{8} k_i(y)^3 = \frac{p(\theta)}{r^3} + O(r^{-4-\sigma}).$$

- There exists a smooth function $h_0(r, \theta)$ such that $h_0 = O(r^{-1})$ and for some $\sigma > 0$,

$$\mathcal{J}_\Gamma[h_0] = \frac{p(\theta)}{r^3} + O(r^{-4-\sigma}) \quad \text{as } r \to +\infty.$$

$$\mathcal{J}_\Gamma(h) := \Delta_\Gamma h + |A_\Gamma|^2 h = c \sum_{i=1}^{8} k_i^3 + O(r^{-4-\sigma}) \quad \text{in } \Gamma.$$

Our final problem then becomes $h = h_0 + h_1$ where

$$h_1 = T(O(r^{-4-\sigma}) + \mathcal{N}(h_0 + h_1))$$

which we can solve for $h_1 = O(r^{-2-\sigma})$, using contraction mapping principle, keeping track of Lipschitz dependence in h of the objects involved in in $\mathcal{N}(h)$.

The Jacobi operator

$$\mathcal{J}_\Gamma[h] = \Delta_\Gamma h + |A_\Gamma(y)|^2 h,$$

is the linearization of the mean curvature, when normal perturbations are considered. In the case of a minimal graph $x_9 = F(x')$, if we linearize along vertical perturbations we get

$$H'(F)[\phi] \; = \; \nabla \cdot \left(\frac{\nabla \phi}{\sqrt{1+|\nabla F|^2}} - \frac{(\nabla F \cdot \nabla \phi)}{(1+|\nabla F|^2)^{\frac{3}{2}}} \nabla F \right).$$

These two operators are linked through the relation

$$\mathcal{J}_\Gamma[h] = H'(F)[\phi], \quad \text{where} \quad \phi(x') = \sqrt{1+|\nabla F(x')|^2}\, h(x', F(x')).$$

The relation $\mathcal{J}_{\Gamma_0}[h] = H'(F_0)[\sqrt{1+|\nabla F_0|^2}\, h]$ also holds.

Next we discuss the proofs of the facts used above:

1. If $g = O(r^{-4-\sigma})$ there is a unique solution to $\mathcal{J}_\Gamma[h] = g$ with

$$\|(1+r)^{2+\sigma}\, h\|_{L^\infty(\Gamma)} \;\leq\; C\, \|(1+r)^{4+\sigma}\, g\|_{L^\infty(\Gamma)}\,.$$

2. There is a function p smooth, with $p(\frac{\pi}{2} - \theta) = -p(\theta)$ for all $\theta \in (0, \frac{\pi}{4})$ such that

$$\sum_{i=1}^{8} k_i(y)^3 = \frac{p(\theta)}{r^3} + O(r^{-4-\sigma}).$$

3. There exists $h_0(r, \theta)$ such that $h_0 = O(r^{-1})$ and

$$\mathcal{J}_\Gamma[h_0] = \frac{p(\theta)}{r^3} + O(r^{-4-\sigma}) \quad \text{as } r \to +\infty.$$

The closeness between $\mathcal{J}_{\Gamma_0}$ and $\mathcal{J}_{\Gamma}$.

Let $p \in \Gamma$ with $r(p) \gg 1$. There is a unique $\pi(p) \in \Gamma_0$ such that $\pi(p) = p + t_p \nu(p)$.
Let us assume

$$\tilde{h}(\pi(y)) = h(y), \quad \text{for all} \quad y \in \Gamma, \quad r(y) > r_0.$$

Then

$$\mathcal{J}_{\Gamma}[h](y) =$$

$$[\mathcal{J}_{\Gamma_0}[h_0] + O(r^{-2-\sigma}) D^2_{\Gamma_0} h_0 + O(r^{-3-\sigma}) D_{\Gamma_0} h_0 + O(r^{-4-\sigma}) h_0]\, (\pi(y))\,.$$

We keep in mind that $\mathcal{J}_{\Gamma_0}[h] = H'(F_0)[\sqrt{1+|\nabla F_0|^2} h]$ and make explicit computations.

We compute explicitly

$$\begin{aligned} H'(F_0)[\phi] \;=\; & \frac{1}{r^7 \sin^3(2\theta)} \Big\{ (9g^2\,\tilde{w} r^3 \phi_\theta)_\theta + (r^5 g'^2\,\tilde{w}\phi_r)_r \\ & - 3(gg'\,\tilde{w} r^4 \phi_r)_\theta - 3(gg'\,\tilde{w} r^4 \phi_\theta)_r \Big\} \\ & + \frac{1}{r^7 \sin^3(2\theta)} \Big\{ (r^{-1}\,\tilde{w}\phi_\theta)_\theta + (r\tilde{w}\phi_r)_r \Big\}, \end{aligned}$$

where

$$\tilde{w}(r,\theta) := \frac{\sin^3 2\theta}{(r^{-4} + 9g^2 + g'^2)^{\frac{3}{2}}}.$$

Further expand

$$L[\phi] := H'(F_0)[\phi] := L_0 + L_1,$$

with

$$\begin{aligned} L_0[\phi] \;=\; & \frac{1}{r^7 \sin^3(2\theta)} \Big\{ (9g^2\, \tilde{w}_0 r^3 \phi_\theta)_\theta + (r^5 g'^2\, \tilde{w}_0 \phi_r)_r \\ & \qquad - 3(gg'\, \tilde{w}_0 r^4 \phi_r)_\theta - 3(gg'\, \tilde{w}_0 r^4 \phi_\theta)_r \Big\} \\ & + \frac{1}{r^7 \sin^3(2\theta)} \Big\{ (r^{-1}\, \tilde{w}_0 \phi_\theta)_\theta + (r \tilde{w}_0 \phi_r)_r \Big\}, \end{aligned}$$

where

$$\tilde{w}_0(\theta) := \frac{\sin^3 2\theta}{(9g^2 + g'^2)^{\frac{3}{2}}}.$$

An important fact: If $0 < \sigma < 1$ there is a positive supersolution $\bar{\phi} = O(r^{-\sigma})$ to

$$-L[\bar{\phi}] \geq \frac{1}{r^{4+\sigma}} \quad \text{in } \Gamma$$

We have that

$$L_0[r^{-\sigma} q(\theta)] = \frac{1}{r^{4+\sigma}} \frac{9g^{\frac{4-\sigma}{3}}}{\sin^3 2\theta} \left[\frac{g^{\frac{2}{3}} \sin^3 2\theta}{(9g^2 + g'^2)^{\frac{3}{2}}} \left(g^{\frac{\sigma}{3}} q \right)' \right]' = \frac{1}{r^{4+\sigma}}.$$

if and only if $q(\theta)$ solves the ODE

$$\left[\frac{g^{\frac{2}{3}} \sin^3 2\theta}{(9g^2 + g'^2)^{\frac{3}{2}}} \left(g^{\frac{\sigma}{3}} q \right)' \right]' = \frac{1}{9} \sin^3 2\theta g(\theta)^{-\frac{4-\sigma}{3}}, \ .$$

A solution in $(\frac{\pi}{4}, \frac{\pi}{2})$:

$$q(\theta) = \frac{1}{9} g^{-\frac{\sigma}{3}}(\theta) \int_{\frac{\pi}{4}}^{\theta} \frac{(\,9g^2 + g'^2\,)^{\frac{3}{2}}}{g^{\frac{2}{3}} \sin^3(2s)}\, ds \int_s^{\frac{\pi}{2}} g^{-\frac{4-\sigma}{3}}(\tau) \sin^3(2\tau)\, d\tau\,.$$

Since $g'(\frac{\pi}{4}) > 0$, q is defined up to $\frac{\pi}{4}$ and can be extended smoothly (evenly) to $(0, \frac{\pi}{4})$. Thus and $\bar{\phi} := q(\theta) r^{-\sigma}$ satisfies

$$-L_0(\bar{\phi}) = r^{-4-\mu} \quad \text{in } \mathbb{R}^8.$$

We can show that also $-L(\bar{\phi}) \geq r^{-4-\sigma}$ for all large r. Thus

$$-\mathcal{J}_{\Gamma_0}[\bar{h}] \geq r^{-4-\sigma}, \quad \bar{h} = \frac{\phi}{\sqrt{1 + |\nabla F_0|^2}} \sim r^{-2-\sigma}$$

The closeness of $\mathcal{J}_\Gamma$ and $\mathcal{J}_{\Gamma_0}$ makes $\bar{h}$ to induce a positive supersolution $\hat{h} \sim r^{-2-\sigma}$ to

$$-\mathcal{J}_\Gamma[\hat{h}] \geq r^{-4-\sigma} \quad \text{in } \Gamma.$$

We conclude by a barrier argument that Fact 1 holds: if $\|(1+r^{4+\sigma})\, g\|_{L^\infty(\Gamma)} < +\infty$ there is a unique h with $\mathcal{J}_\Gamma[h] = \mathrm{g}$ and

$$\|(1+r)^{2+\sigma}\, h\|_{L^\infty(\Gamma)} \;\leq\; C\, \|(1+r)^{4+\sigma}\, \mathrm{g}\|_{L^\infty(\Gamma)}\,.$$

Let $k_i^0(y)$ be the principal curvatures of Γ_0.

The following hold:

- $$\sum_{i=1}^{8} k_i(y)^3 = \sum_{i=1}^{8} k_i^0(\pi(y))^3 + O(r^{-4-\sigma})$$

- $$\sum_{i=1}^{8} k_i^0(y)^3 = \frac{p(\theta)}{r^3} + O(r^{-4-\sigma})$$

p smooth, $p(\frac{\pi}{2} - \theta) = -p(\theta)$ for all $\theta \in (0, \frac{\pi}{4})$.

We claim: *there exists a smooth function* $h_*(r, \theta)$ *such that* $h_* = O(r^{-1})$ *and for some* $\sigma > 0$,

$$\mathcal{J}_{\Gamma_0}[h_*] = \frac{p(\theta)}{r^3} + O(r^{-4-\sigma}) \quad \text{as } r \to +\infty.$$

Setting $h_0(y) = h_*(\pi(y))$ we then get $h_0 = O(r^{-1})$ and

$$\mathcal{J}_\Gamma(h) := \Delta_\Gamma h + |A_\Gamma|^2 h \; = \; c \sum_{i=1}^{8} k_i^3 + O(r^{-4-\sigma}) \quad \text{in } \Gamma,$$

namely the validity of Fact 2.

Construction of h_*.
We argue as before (separation of variables) to find $q(\theta)$ solution of

$$L_0(r\, q(\theta)) = \frac{p(\theta)}{r^3}, \theta \in (\frac{\pi}{4}, \frac{\pi}{2}).$$

$$q(\theta) = -\frac{1}{9} g^{\frac{1}{3}}(\theta) \int_{\frac{\pi}{4}}^{\theta} (\, 9g^2 + g'^2\,)^{\frac{3}{2}} \, \frac{g^{-\frac{2}{3}} ds}{\sin^3(2s)} \int_s^{\frac{\pi}{2}} p(\tau) g^{-\frac{5}{3}}(\tau) \sin^3(2\tau)\, d\tau \, .$$

Let $\eta(s) = 1$ for $s < 1$, $= 0$ for $s > 2$ be a smooth cut-off function. Then

$$\phi_0(r,\theta) \;:=\; (1-\eta(s))\, r\, q(\theta) \quad \text{in } (\frac{\pi}{4},\frac{\pi}{2}), \quad s = r^2 g(\theta).$$

satisfies

$$L(\phi_0) = \frac{p(\theta)}{r^3} + O(r^{-4-\frac{1}{3}}).$$

Finally, the function

$$h_* = \frac{\phi_0}{\sqrt{1+|\nabla F_0|^2}} = O(r^{-1})$$

extended oddly through $\theta = \frac{\pi}{4}$ satisfies

$$J_{\Gamma_0}[h_*] = \frac{p(\theta)}{r^3} + O(r^{-4-\frac{1}{3}})$$

as desired.

Loosely speaking: The method described above applies to find an entire solution u_ε to $\Delta u + u - u^3 = 0$ with transition set near $\Gamma_\varepsilon = \varepsilon^{-1}\Gamma$ whenever Γ is a minimal hypersurface in $\mathbb{R}^N$, that splits the space into two components, and for which enough control at infinity is present to invert globally its Jacobi operator.

An important example for $N = 3$: finite Morse index solutions.

Theorem (D., Kowalczyk, Wei (2009))

Let Γ be a complete, embedded minimal surface in $\mathbb{R}^3$ with finite total curvature: $\int_\Gamma |K| < \infty$*,*K *Gauss curvature.*

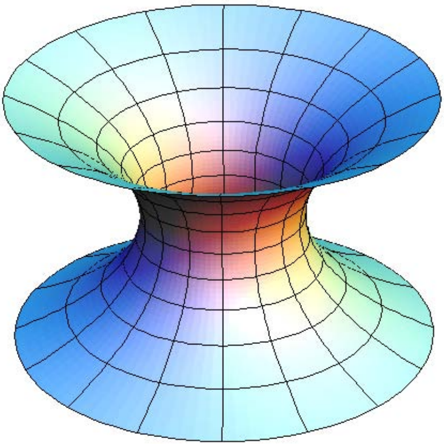
If Γ is non-degenerate, namely its bounded Jacobi fields originate only from rigid motions, then for small $\varepsilon > 0$ there is a solution u_ε to (AC) with

$$u_\varepsilon(x) \approx w(t), \quad x = y + t\nu_\varepsilon(y).$$

In addition $i(u_\varepsilon) = i(\Gamma)$ *where i denotes Morse index.*

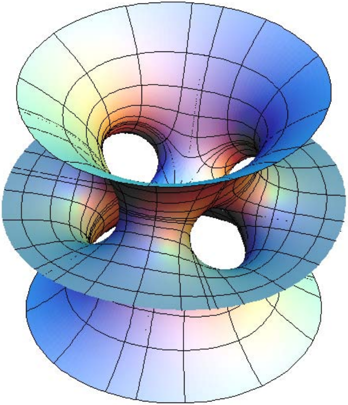
Examples: nondegeneracy and Morse index are known for the catenoid and Costa-Hoffmann-Meeks surfaces (Nayatani (1990), Morabito, (2008)).

$\Gamma =$ a catenoid: $\quad \exists\ u_\varepsilon(x) = w(\zeta) + O(\varepsilon),\ x = y + t\nu_\varepsilon(y).$



u_ε axially symmetric: $u_\varepsilon(x) = u_\varepsilon(\sqrt{x_1^2 + x_2^2}\,, x_3)$, x_3 rotation axis coordinate. $i(u_\varepsilon) = 1$

Γ = CHM surface genus $\ell \geq 1$:



$\exists\ u_\varepsilon(x) = w(\zeta) + O(\varepsilon)$, $x = y + \zeta\nu_\varepsilon(y)$. $i(u_\varepsilon) = 2\ell + 3$.

• Nondegeneracy: The only nontrivial bounded solutions of

$$\mathcal{J}_\Gamma(\phi) = \Delta_\Gamma \phi - 2K\phi = 0$$

arise from translations and rotation about the common symmetry axis (x_3) of the ends: $\nu_i(x)$ $i = 1, 2, 3,$ $x_2\nu_1(x) - x_1\nu_2(x)$.

• $i(\Gamma)$, the Morse index of Γ, is the number of negative eigenvalues of J_Γ in $L^\infty(\Gamma)$. This number is finite $\iff$ Γ has finite total curvature.

• $i(\Gamma) = 0$ for the plane, $= 1$ for the catenoid and $= 2\ell + 3$ for the CHM surface genus ℓ.

Morse index of a solution u of (AC), $i(u)$: roughly, the number of negative eigenvalues of the linearized operator, namely those of the problem

$$\Delta\phi + (1 - 3u^2)\phi + \lambda\phi = 0 \quad \phi \in L^\infty(\mathbb{R}^N).$$

De Giorgi solution: "stable", $i(u) = 0$ since $\lambda = 0$ is an eigenvalue with eigenfunction $\partial_{x_N} u > 0$.

$i(u) = 0 \implies$ DG statement for $N = 2$ (Dancer). Open if $N \geq 3$.

A recent result: $i(u) = 0$ **does not imply** DG statement for $N = 8$ (Example by Pacard and Wei).

Another application of the BDG minimal graph: Overdetermined semilinear equation

Ω smooth domain, f Lipschitz

$$\Delta u + f(u) = 0, \;\; u > 0 \quad \text{in } \Omega, \;\; u \in L^\infty(\Omega) \qquad (S)$$

$$u = 0, \quad \partial_\nu u = constant \quad \text{on } \partial\Omega$$

Let us assume that (S) is solvable. What can we say about the geometry of Ω?

Serrin (1971) proved that if Ω is **bounded** and there is a solution to (S) then Ω must be a ball.

We consider the case of an entire *epigraph*

$$\Omega = \{(x', x_N) \; / \; x' \in \mathbb{R}^{N-1}, \; x_N > \varphi(x')\}, \quad \Gamma = \partial\Omega.$$

$$\Omega = \{(x', x_N) \ / \ x' \in \mathbb{R}^{N-1}, \ x_N > \varphi(x')\}, \quad \Gamma = \partial\Omega.$$

- Berestycki, Caffarelli and Nirenberg (1997) proved that if φ is Lispchitz and *asymptotically flat* then it must be linear and u depends on only one variable. They conjecture that this should be true for any arbitrary smooth function φ.
- Farina and Valdinoci (2009) lifted asymptotic flatness for $N = 2, 3$ and for $N = 4, 5$ and $f(u) = u - u^3$.

Theorem (D., Pacard, Wei (2010))

In Dimension $N \geq 9$ there exists a solution to Problem (S) with $f(u) = u - u^3$, in an entire epigraph Ω which is not a half-space.

The proof consists of finding the region Ω in the form

$$\partial\Omega = \{y + \varepsilon h(\varepsilon y)\nu(\varepsilon y) \ / \ y \in \Gamma_\varepsilon\}.$$

for h a small decaying function on Γ. Here Γ is a BDG graph. We set

$$u_0(x) = w(t), \quad x = y + (t + \varepsilon h(\varepsilon y))\nu(\varepsilon y) \quad \Omega = \{t > 0\}.$$

At main order ϕ should satisfy

$$\partial_{tt}\phi + \Delta_{\Gamma_\varepsilon}\phi + f'(w(t))\phi = E$$

$$\phi(0,y) = 0, \phi_t(0,y) =\approx \alpha\varepsilon = constant$$

$$E = \Delta u_0 + f(u_0) \ =$$

$$\varepsilon^4 |\nabla_{\Gamma\varepsilon\zeta} h(\varepsilon y)|^2 w''(t) - [\varepsilon^3 \Delta_{\Gamma\varepsilon\zeta} h(\varepsilon y) \ + \varepsilon H_{\Gamma\varepsilon\zeta}(\varepsilon y)] \, w'(t),$$

$$E = \varepsilon H_\Gamma(\varepsilon y)\, w'(t) + O(\varepsilon^2)$$

The construction carries over for regions whose boundaries are more general surfaces.

Let us assume, more generally that Γ is a smooth surface such that

$$H_\Gamma \equiv H = constant$$

Namely Γ is a constant mean curvature surface.

For $x = y + \varepsilon(t + \varepsilon h(\varepsilon y))$, we look now for a solution for $t > 0$ with

$$u(t, y) = w(t) + \phi(t, y), \quad \phi(0, y) = 0.$$

Imposing $\alpha = (H/w'(0)) \int_0^\infty w'(t)^2 dt$. we can solve

$$\psi'' + f'(w(t))\psi = Hw'(t), \quad t > 0, \quad \psi(0) = 0, \psi'(0) = \alpha$$

which is solvable for ψ bounded. Then the approximation $u_1(x) = w(t) + \varepsilon\psi(t)$ produces a new error of order ε^2. And the equation for $\phi = \varepsilon\psi(t) + \phi_1$ now becomes

$$\partial_{tt}\phi_1 + \Delta_{\Gamma_\varepsilon}\phi_1 + f'(w(t))\phi_1 = E_1 = O(\varepsilon^2)$$

$$\phi_1(0, y) = 0, \phi_{1,t}(0, y) = 0$$

The construction follows a scheme similar to that for the entire solution, but it is more subtle in both theories needed in Steps 1 and 2.

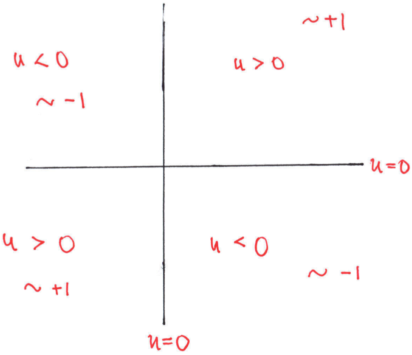
The case $N = 2$: Very few solutions known with $1 \leq i(u) < +\infty$.

• Dang, Fife, Peletier (1992). The cross saddle solution: $u(x_1, x_2) > 0$ for $x_1, x_2 > 0$,

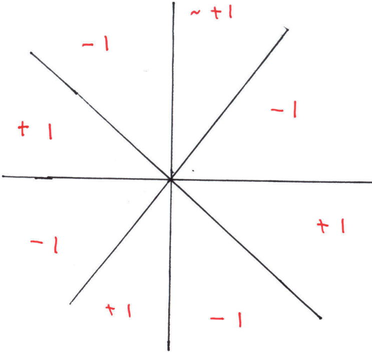
$$u(x_1, x_2) = -u(-x_1, x_2) = -u(x_1, -x_2).$$

Nodal set two lines (4 ends). Super-subsolutions in first quadrant.

• Alessio, Calamai, Montecchiari (2007). Extension: saddle solution with dihedral symmetry. Nodal set k lines ($2k$ ends), $k \geq 2$. Presumably $i(u) = k - 1$.



DANG-FIFE-PELETIER 92
(4-END)



ALESSIO-CALAMAI-MONTECCHIARI 07
(2k-END)

The saddle solutions

A result: Existence of entire solutions with embedded level set and finite number of transition lines of $\Delta u + u - u^3 = 0$ in $\mathbb{R}^2$:

Solutions with k "nearly parallel" transition lines are found for any $k \geq 1$.

Theorem (del Pino, Kowalczyk, Pacard, Wei (2007))

If f satisfies

$$\frac{\sqrt{2}}{24} f''(z) = e^{-2\sqrt{2}f(z)}, \quad f'(0) = 0,$$

and $f_\varepsilon(z) := \sqrt{2}\log\frac{1}{\varepsilon} + f(\varepsilon z)$, *then there exists a solution* u_ε *to (AC) in* $\mathbb{R}^2$ *with*

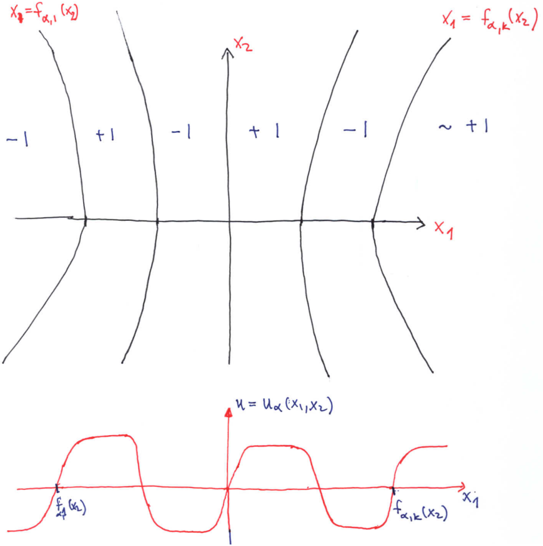
$$\boxed{u_\varepsilon(x_1, x_2) = w(x_1 + f_\varepsilon(x_2)) + w(x_1 - f_\varepsilon(x_2)\ - 1 + o(1)}$$

as $\varepsilon \to 0^+$. *Here* $w(s) = \tanh(s/\sqrt{2})$.

This solution has 2 transition lines.

$$f(z) = A|z| + B + o(1) \quad \text{as } z \to \pm\infty.$$

More in general: the equilibrium of k far-apart, embedded transition lines is governed by the **Toda system,** a classical integrable model for scattering of particles on a line under the action of a repulsive exponential potential:



$$u_\varepsilon(x_1, x_2) = \sum_{j=1}^{k} (-1)^{j-1} w(x_1 - f_{\varepsilon,j}(x_2)) - \frac{1}{2}(1 + (-1)^k) + o(1)$$

The Toda system:

$$\frac{\sqrt{2}}{24} f_j'' = e^{-\sqrt{2}(f_j - f_{j-1})} - e^{-\sqrt{2}(f_{j+1} - f_j)}, \quad j = 1, \ldots k,$$

$f_0 \equiv -\infty, \ f_{k+1} \equiv +\infty.$

Given a solution f (with asymptotically linear components), if we scale

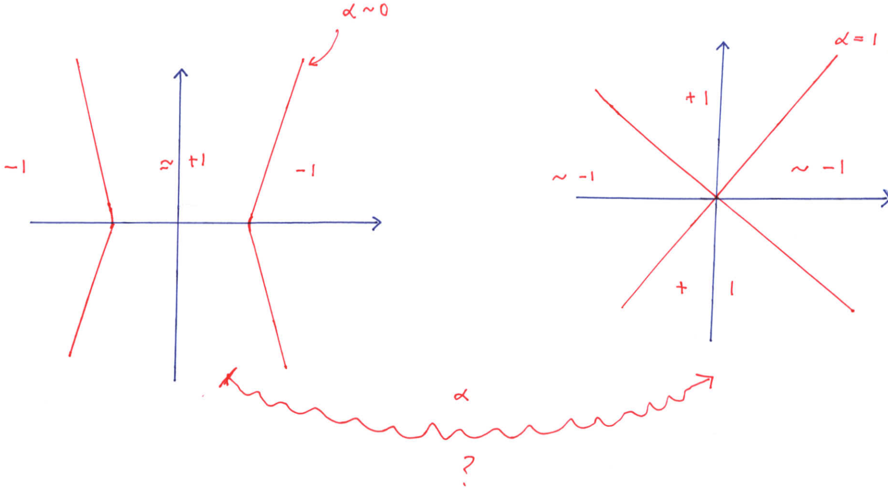
$$f_{\varepsilon,j}(z) \ := \ \sqrt{2}\,(j - \frac{k+1}{2})\log\frac{1}{\varepsilon} + f_j(\varepsilon z),$$

then there is a solution with k transitions:

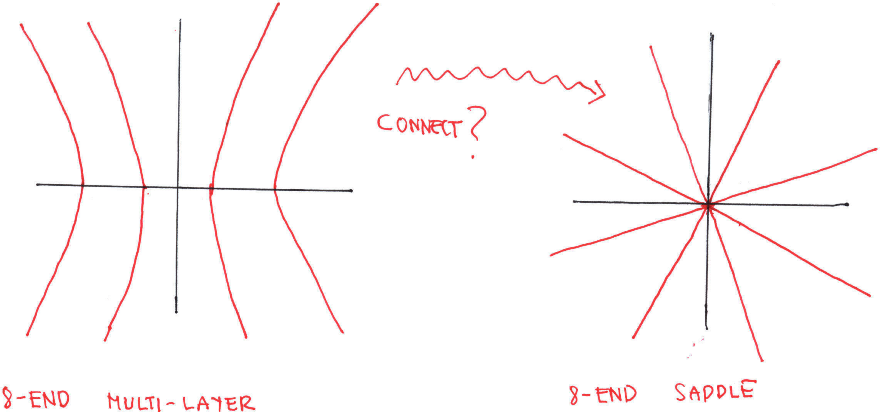
$$u_\varepsilon(x_1, x_2) = \sum_{j=1}^{k}(-1)^{j-1} w(x_1 - f_{\varepsilon,j}(x_2)\,) \ - \ \frac{1}{2}(1 + (-1)^k) \ + \ o(1)$$

- Pacard and Ritoré (2002) found a solution with a transition layer across a nondegenerate minimal submanifold of codimension 1 in a compact manifold.
- Kowalczyk (2002) found such a solution associated to a nondegenerate segment of a planar domain, with Neumann boundary conditions. D., Kowalczyk, Wei (2005) found multiple interfaces in that setting, with equilibrium driven by the Toda system.
- We believe the nodal set of any finite Morse index solutions in $\mathbb{R}^2$ must be asymptotic to an even, finite number of rays.

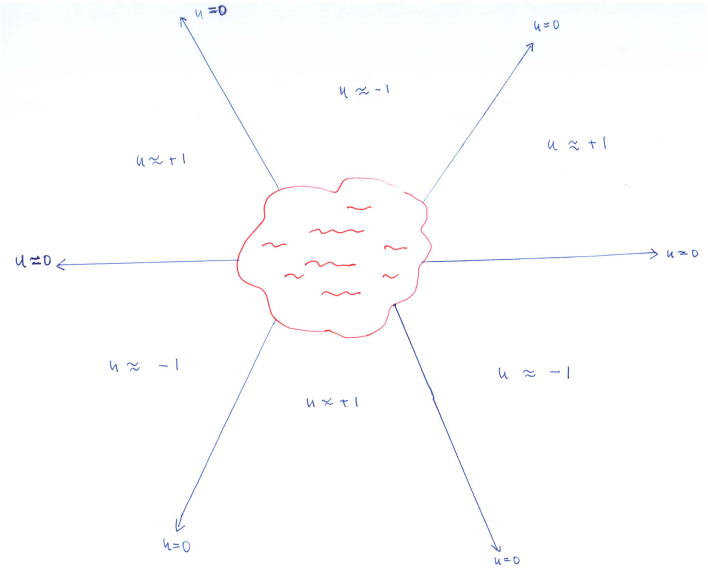
We conjecture: The 4-end (two-line) solution is a limit case of a continuum of solutions with Morse index 1 that has the cross saddle as the other endpoint *All intermediate slopes missing*. This is also the case for $k > 2$.



2-line transition layer and 4 end saddle: Do they connect?



Do they connect?



General $2k$-end

Some evidence:

A result: (D., Kowalczyk, Pacard) *given a* nondegenerate $2k$*-end solution* u*, the class of all* $2k$*-end solutions nearby constitutes a* $2k$*-dimensional manifold.*

This is the case for the solution with k nearly parallel transition lines and the cross saddle (Kowalczyk, Liu (2009)). For 2 transition lines we thus have one parameter (ε) besides translations and rotations.