Problema Extra (Es importante!)

Consideremos el sistema de coordenadas dado por:

$$\vec{r}(x,\rho,\theta) = (x, \rho\cos\theta, \rho\,\text{sen}\,\theta) \qquad x\in\mathbb{R},\ \theta\in[0,2\pi),\ \rho\geq 0.$$

a) Determinar el triedro de vectores unitarios $\hat{x}, \hat{\rho}, \hat{\theta}$.
¿Son ortogonales?, Calcule $\hat{\theta}\times\hat{x}$, $\hat{\theta}\times\hat{\rho}$.

Sol. Recordemos que si tenemos un sist. de coord. dado por $\vec{r}=\vec{r}(u,v,w)$
$\Rightarrow$ los vect. unit se calculan vía $\hat{u} = \dfrac{\partial\vec{r}/\partial u}{\underbrace{\|\partial\vec{r}/\partial u\|}}$ $\leftarrow h_u$ : factor escalar.

En este caso:

$$\hat{x} = \frac{\frac{\partial\vec{r}}{\partial x}}{\left\|\frac{\partial\vec{r}}{\partial x}\right\|} = \frac{(1,0,0)}{\|(1,0,0)\|} = (1,0,0) \qquad \hat{\rho} = \frac{\frac{\partial\vec{r}}{\partial\rho}}{\left\|\frac{\partial\vec{r}}{\partial\rho}\right\|} = \frac{(0,\cos\theta,\text{sen}\,\theta)}{\underbrace{\|(0,\cos\theta,\text{sen}\,\theta)\|}_{=1}} = (0,\cos\theta,\text{sen}\,\theta)$$

$$\hat{\theta} = \frac{\partial\vec{r}/\partial\theta}{\|\partial\vec{r}/\partial\theta\|} = \frac{(0,-\rho\,\text{sen}\,\theta,\rho\cos\theta)}{\|(0,-\rho\,\text{sen}\,\theta,\rho\cos\theta)\|} = \frac{(0,-\rho\,\text{sen}\,\theta,\rho\cos\theta)}{\sqrt{\rho^2(\text{sen}^2\theta+\cos^2\theta)}} = \frac{(0,-\rho\,\text{sen}\,\theta,\rho\cos\theta)}{\rho}$$

$$= (0,-\text{sen}\,\theta,\cos\theta)$$

además:

$$\hat{x}\cdot\hat{\rho} = (1,0,0)\cdot(0,\cos\theta,\text{sen}\,\theta) = (0,0,0)$$

$$\hat{x}\cdot\hat{\theta} = (1,0,0)\cdot(0,-\text{sen}\,\theta,\cos\theta) = (0,0,0)$$

$$\hat{\rho}\cdot\hat{\theta} = (0,-\text{sen}\,\theta,\cos\theta)\cdot(0,\cos\theta,\text{sen}\,\theta)$$

$$= -\text{sen}\,\theta\cos\theta + \text{sen}\,\theta\cos\theta = 0.$$

i.e. son ortogonales.

$$\hat{\theta}\times\hat{x} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & -\text{sen}\,\theta & \cos\theta \\ 1 & 0 & 0 \end{vmatrix} = \hat{\imath}(0) + \hat{\jmath}(\cos\theta) + \hat{k}(+\text{sen}\,\theta)$$

$$= (0,\cos\theta,\text{sen}\,\theta) = \hat{\rho}.$$

$$\hat{\theta} \times \hat{\rho} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & -\operatorname{sen}\theta & \cos\theta \\ 0 & \cos\theta & \operatorname{sen}\theta \end{vmatrix} = \hat{\imath}\underbrace{(-\operatorname{sen}^2\theta - \cos^2\theta)}_{-1} + \hat{\jmath}(0) + \hat{k}(0)$$

$$= -\hat{\imath} = -\hat{x}.$$

De esto, podemos notar que el orden (para regla de mano derecha) es:

$$\hat{\theta} \to \hat{x} \to \hat{\rho} \quad \text{i.e.} \quad \hat{\theta} \times \hat{x} = \hat{\rho}, \quad \hat{\rho} \times \hat{\theta} = \hat{x}$$

$$\hat{x} \times \hat{\rho} = \hat{\theta}.$$

b) Determinar expresiones para gradiente, divergencia, laplaciano, rotor en este sistema de coordenadas.

gradiente $\nabla f = \dfrac{1}{h_u}\dfrac{\partial f}{\partial u}\hat{u} + \dfrac{1}{h_v}\dfrac{\partial f}{\partial v}\hat{v} + \dfrac{1}{h_w}\dfrac{\partial f}{\partial w}\hat{w}$

$$\Rightarrow \nabla f = \frac{1}{1}\frac{\partial f}{\partial x}\hat{x} + \frac{1}{\rho}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{1}\frac{\partial f}{\partial \rho}\hat{\rho}.$$

divergencia: $\operatorname{div}\vec{F} = \dfrac{1}{h_u h_v h_w}\left[\dfrac{\partial}{\partial u}(F_u h_v h_w) + \dfrac{\partial}{\partial v}(F_v h_u h_w) + \dfrac{\partial}{\partial w}(h_u h_v F_w)\right]$

$$\Rightarrow \operatorname{div}\vec{F} = \frac{1}{\rho}\left[\frac{\partial}{\partial x}(F_x \cdot \rho) + \frac{\partial}{\partial \theta}[F_\theta \cdot 1] + \frac{\partial}{\partial \rho}(F_\rho \cdot \rho)\right]$$

Laplaciano: $\Delta f = \nabla^2 f = \dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}\left(\rho\dfrac{\partial f}{\partial \rho}\right) + \dfrac{1}{\rho^2}\dfrac{\partial^2 f}{\partial \theta^2} + \dfrac{\partial^2 f}{\partial x^2}.$

rotor: $\operatorname{rot}\vec{F} = \dfrac{1}{h_v h_w}\left[\dfrac{\partial}{\partial v}(F_w h_w) - \dfrac{\partial}{\partial w}(F_v h_v)\right]\hat{u} + \dfrac{1}{h_u h_w}\left[\dfrac{\partial}{\partial w}F_u h_u - \dfrac{\partial}{\partial u}F_w h_w\right]\hat{v}$

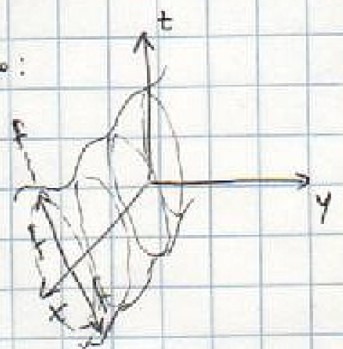
$$+ \frac{1}{h_u h_v}\left[\frac{\partial}{\partial u}(F_v h_v) - \frac{\partial}{\partial v}(F_u h_u)\right]\hat{w}$$

$$\Rightarrow \operatorname{rot}\vec{F} = \frac{1}{\rho}\left[\frac{\partial F_x}{\partial \theta} - \rho\frac{\partial F_\theta}{\partial x}\right]\hat{\rho} + \left[\frac{\partial F_\rho}{\partial x} - \frac{\partial F_x}{\partial \rho}\right]\hat{\theta} + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho F_\theta) - \frac{\partial F_\rho}{\partial \theta}\right]\hat{i}.$$

(verificar).

c) $f$ no negativa y dif. tq $f:[a,b] \to \mathbb{R}^+$
los puntos $y^2+z^2 = f(x)^2$. Verif. que se puede param. como
$\vec{r}_1(x,\theta) = x\hat{i} + f(x)\hat{\rho}$.

Grafico:




$y^2+z^2 = f^2(x)$
↳ circulos en el plano $y, z$ de radio $f(x)$.

← superficie de revolución.

notar que si usamos estas coord:

$y = \rho\cos\theta$, $z = \rho\,\text{sen}\,\theta \Rightarrow y^2+z^2 = \rho^2$.

$\therefore f(x)^2 = \rho^2 \underset{f\geq 0}{\Rightarrow} f(x) = \rho$

Así, En este caso: $\vec{r}(x,\theta) = x\hat{i} + \rho\hat{\rho} \overset{\text{Este caso!}}{=} x\hat{i} + f(x)\hat{\rho}$ //