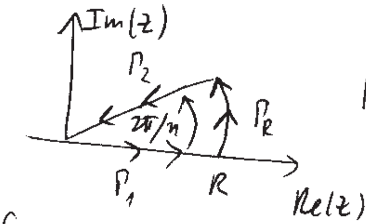
b) Si $n \geq m+2$

$$\int_0^\infty \frac{x^m}{x^n+1}\,dx = \frac{\pi}{n\,\operatorname{sen}\left(\frac{\pi}{n}(m+1)\right)}$$

Hint: (Si, lo daban) Considere el camino:



$\Gamma = \Gamma_1 \cup \Gamma_R \cup \Gamma_2$.

Sol. Sea $f(z) = \dfrac{z^m}{z^n+1}$. Calculemos $\displaystyle\int_\Gamma f(z)dz = \int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz + \int_{\Gamma_R} f(z)dz$

Por Teo Res: $\displaystyle\int_\Gamma f(z)dz = 2\pi i \sum_{p \text{ es polo}} \operatorname{Res}(f,p)\,\operatorname{ind}_\Gamma(p)$

los polos de f: $z^n+1=0 \iff z^n=-1$ ie. raíces n-ésimas de -1.

[Rcdo Raíces n-ésimas de un complejo cualquiera:

$z^n = w$, $w\in\mathbb{C}$ $\Rightarrow$ Si $z=|z|e^{i\theta} \Rightarrow z^n = w \iff |z|^n e^{in\theta} = |w|e^{i\varphi}$

$\Rightarrow |z| = \sqrt[n]{|w|} \wedge n\theta = \varphi + 2k\pi \Rightarrow \theta_k = \dfrac{\varphi+2k\pi}{n}$

$\therefore$ las n raíces de w: $z_k = \sqrt[n]{|w|}\cdot e^{i\left(\frac{\varphi+2k\pi}{n}\right)}$ $k\in\{0,\dots,n-1\}$.]

En nuestro caso: $w=-1 = 1\cdot e^{i\pi} \Rightarrow \varphi=\pi$. $\Rightarrow$ las n raíces:
$|w|=1$ $\qquad z_k = e^{i\left(\frac{\pi+2k\pi}{n}\right)}$ $k\in\{0,\dots,n-1\}$

Para $R>1$ solo $z_0 = e^{i\pi/n}$ está ~~dentro del~~ encerrado por el camino

(pues $z_k = e^{i\left(\frac{\pi+2k\pi}{n}\right)}$ es tq $\arg(z_k) = \frac{\pi+2k\pi}{n} > \frac{2\pi}{n}$ si $k>0$.)

$\therefore \displaystyle\int_\Gamma f(z)\,dz = 2\pi i \operatorname{Res}(f, e^{i\pi/n})$.

orden del polo es 1 pues $\overset{=P(z)}{z^n+1}$ tiene n raíces distintas.

forma 0/0

$$\operatorname{Res}(f, e^{i\pi/n}) = \lim_{z\to e^{i\pi/n}} (z-e^{i\pi/n})\cdot\frac{z^m}{z^n+1} = \lim_{z\to e^{i\pi/n}} z^m \cdot \lim_{z\to e^{i\pi/n}} \frac{(z-e^{i\pi/n})}{z^n+1}$$

$$\mathrm{Res}(f, e^{i\pi/n}) = \lim_{z\to e^{i\pi/n}} z^m \cdot \lim_{z\to e^{i\pi/n}} \frac{(z-e^{i\pi/n})}{z^n+1}$$) forma $0/0$, podemos usar L'Hôpital

$$= \lim_{z\to e^{i\pi/n}} z^m \cdot \lim_{z\to e^{i\pi/n}} \frac{1}{n z^{n-1}} = e^{i\pi\frac{m}{n}} \cdot \frac{1}{n\cdot e^{i\pi\frac{(n-1)}{n}}} = \frac{e^{i\pi\frac{m}{n}}}{n e^{i\pi(\frac{n-1}{n})}}$$

$$= \frac{e^{i\pi m/n}}{n e^{i\pi(1-\frac{1}{n})}} = \frac{e^{i\pi\frac{m}{n}}}{n \underbrace{e^{i\pi}}_{-1} \cdot e^{-i\pi/n}} = \frac{-e^{i\pi\frac{m}{n}}}{n e^{-i\frac{\pi}{n}}} = \frac{-e^{i\pi\frac{(m+1)}{n}}}{n}$$

$\therefore$ $$\int_\Gamma f(z)dz = -\frac{2\pi i}{n} e^{-i\pi(\frac{m+1}{n})} \qquad (R>1)$$

Veamos las integrales en cada camino (y como se comportan si $R\to\infty$)

Γ_1 $$\int_{\Gamma_1} f(z)dz \underset{\substack{\uparrow\\ \vec{r}(t)=t\\ t\in[0,R]}}{=} \int_0^R f(t)dt = \int_0^R \frac{t^m}{t^n+1}dt \underset{\uparrow}{=} \int_0^R \frac{x^m}{x^n+1}dx \xrightarrow[R\to\infty]{} \int_0^\infty \frac{x^m}{x^n+1}dx =: I$$

t es variable real (muda!)

Γ_2 $$\int_{\Gamma_2} f(z)dz \underset{\substack{\uparrow\\ \vec{r}(t)=te^{2\pi/n\, i}}}{=} \int_R^0 \frac{t^m e^{i2\pi m/n}}{t^n \underbrace{e^{i2\pi}}_{1}+1} \cdot e^{2\pi/n\, i}dt = -\int_0^R \frac{t^m e^{i2\pi(\frac{m+1}{n})}}{t^n+1}dt.$$

$t\in[0,R]$
(desde R a 0).
$\vec{r}'(t) = e^{2\pi/n\, i}$

$$= -e^{i2\pi(\frac{m+1}{n})}\int_0^R \frac{t^m}{t^n+1}dt \xrightarrow[R\to\infty]{} -e^{-i2\pi(\frac{m+1}{n})} \cdot I.$$

Γ_R $$\int_{\Gamma_R} f(z)dz \underset{\substack{\uparrow\\ \vec{r}(\theta)=Re^{i\theta}}}{=} \int_0^{2\pi/n} \frac{R^m e^{i\theta m}}{R^n e^{i\theta n}+1}\cdot iRe^{i\theta}d\theta = i\int_0^{2\pi/n} \frac{R^{m+1}e^{i\theta(m+1)}}{R^n e^{i\theta n}+1}d\theta$$

$\theta\in[0,2\pi/n]$
$\vec{r}'(\theta) = iRe^{i\theta}$

probemos que si $R\to\infty \Rightarrow \int_{\Gamma_R} f(z)dz \to 0$

En efecto:

$$\left|\int_{\Gamma_R} f(z)\,dz\right| = \left| i \int_0^{2\pi/n} \frac{R^{m+1} e^{i\theta(m+1)}}{R^n e^{i\theta n} + 1}\, d\theta \right| \le \int_0^{2\pi/n} \frac{R^{m+1}\overbrace{|e^{i\theta(m+1)}|}^{=1}}{R^n \underbrace{|e^{i\theta n}|}_{=1}}\, d\theta = R^{m+1-n}\int_0^{2\pi/n} d\theta$$

$$= R^{m+1-n}\cdot\frac{2\pi}{n}$$

como $n \geq m+2 \Rightarrow m+1-n < 0$.

ie. $m+1-n = -\alpha \quad \alpha > 0$.

$$= R^{-\alpha}\cdot\frac{2\pi}{n}$$

$$= \frac{1}{R^{\alpha}}\cdot\frac{2\pi}{n}$$

así: $\displaystyle\int_{\Gamma_R} f(z)\,dz \xrightarrow[R\to\infty]{} 0$

$\xrightarrow[R\to\infty]{} 0$ pues $\alpha > 0$.

así, como
$$\int_{\Gamma} f(z)\,dz = \int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz + \int_{\Gamma_R} f(z)\,dz = -\frac{2\pi i}{n}\, e^{-i\pi\left(\frac{m+1}{n}\right)} \quad \forall R > 1.$$

Tomando $R\to\infty \Rightarrow$
$$I - e^{i2\pi\left(\frac{m+1}{n}\right)} I + 0 = -\frac{2\pi i}{n}\, e^{-i\pi\left(\frac{m+1}{n}\right)}$$

luego
$$I\left(1 - e^{i2\pi\left(\frac{m+1}{n}\right)}\right) = -\frac{2\pi i}{n}\, e^{+i\pi\left(\frac{m+1}{n}\right)}$$

$$\Rightarrow I = \frac{-\frac{2\pi i}{n}\, e^{+i\pi\left(\frac{m+1}{n}\right)}}{1 - e^{i2\pi\left(\frac{m+1}{n}\right)}} = -\frac{2\pi i}{n}\,\frac{1}{\left(\frac{1 - e^{i2\pi\left(\frac{m+1}{n}\right)}}{e^{+i\pi\left(\frac{m+1}{n}\right)}}\right)}$$

Si $\pi\left(\frac{m+1}{n}\right) = u \Rightarrow$
$$I = -\frac{2\pi i}{n}\cdot\frac{1}{\frac{1 - e^{2ui}}{e^{+ui}}} = -\frac{2\pi i}{n}\cdot\frac{1}{e^{-ui} - e^{ui}} = \frac{\ominus 2\pi i}{n}\,\frac{1}{\ominus(e^{ui} - e^{-ui})}$$

$$= \frac{2\pi i}{n}\,\frac{1}{e^{ui} - e^{-ui}} = \frac{\pi}{n}\cdot\frac{1}{\frac{e^{iu} - e^{-iu}}{2i}} = \frac{\pi}{n}\cdot\frac{1}{\operatorname{sen}(u)}$$

$$\therefore\ I = \frac{\pi}{n}\,\frac{1}{\operatorname{sen}\left(\frac{\pi}{n}(m+1)\right)}$$