PROBLEMAS EXTRA CONTROL 2

ma1002 – 2010

sbh

1 (Primitivas o Antiderivadas)

Definición 1.1 *Sean $f\colon A \subseteq \mathbb{R} \to \mathbb{R}$, $I \subseteq A$ un intervalo. Diremos que la función $F\colon I \to \mathbb{R}$ es una primitiva de f en I ssi*

1. *F es continua en I y diferenciable en $\mathrm{Int}(I)$.*
2. *$\forall x \in \mathrm{Int}(I)$, $F'(x) = f(x)$.*

1.1 Tabla de primitivas elementales

- $\int x^n = \frac{x^{n+1}}{n+1} + C$
- $\int \frac{1}{x} = \ln|x| + C$
- $\int \operatorname{sen}(ax) = -\frac{1}{a}\cos(ax) + C$
- $\int \cos(ax) = \frac{1}{a}\operatorname{sen}(ax) + C$
- $\int \tan(ax) = -\frac{1}{a}\ln\cos(ax) + C$
- $\int \sec^2(ax) = \frac{1}{a}\tan(ax) + C$
- $\int \csc^2(ax) = -\frac{1}{a}\cot(ax) + C$
- $\int e^{ax} = ae^{ax} + C$
- $\int \operatorname{senh}(ax) = \frac{1}{a}\cosh(ax) + C$
- $\int \cosh(ax) = \frac{1}{a}\operatorname{senh}(ax) + C$
- $\int \frac{1}{1+x^2} = \arctan(x) + C$
- $\int \frac{1}{\sqrt{1-x^2}} = -\arccos(x) + C$
- $\int \frac{1}{1+x^2} = \arctan(x) + C$
- $\int \frac{1}{1-x^2} = \arg\tanh(x) + C$
- $\int \frac{1}{\sqrt{x^2-1}} = \arg\cosh(x) + C$
- $\int \frac{1}{\sqrt{1+x^2}} = \arg\operatorname{senh}(x) + C$
- $\int \frac{x}{\sqrt{1+x^2}} = \sqrt{1+x^2} + C$

1.2 Propiedades

1. $\int(f+g) = \int f + \int g$
2. $\int(\lambda f) = \lambda \int f$
3. $\int f(u) = \int (f \circ g) \cdot g'(x) + C$ *Fórmula del cambio de variables*
4. $\int f' \cdot g = f \cdot g - \int g' \cdot f$ *Fórmula de integración por partes*

Cambio de notación:

$$\int f := \int f(x)dx$$

Donde x es la variable con respecto a la cual estamos integrando.

Problema 0

Calcule las siguientes primitivas

1. $\int \frac{\operatorname{sen} x}{1+\cos^2 x}$
2. $\int \frac{e^{\arctan x}}{1+x^2}$
3. $\int (ax+b)^n$
4. $\int \frac{f'(x)}{f(x)}$
5. $\int \frac{1}{x^2+2x-3}$

1. Sea $u = \cos x \ \Rightarrow u' = -\operatorname{sen} x$ luego

$$I = \int \frac{\operatorname{sen} x}{1+\cos^2 x} = -\int \frac{u'(x)}{1+u^2(x)}$$

por teorema de cambio de variable

$$I = -\left(\int \frac{1}{1+u^2(x)}\right)_{u=\cos x} = -\arctan u = -\arctan(\cos x) + C$$

2.

$$\begin{aligned} \int \frac{e^{\arctan x}}{1+x^2} &= \int e^{u(x)} u'(x) \\ &= \left(\int e^u\right)_{u=\arctan x} \\ &= e^{\arctan x} + C \end{aligned}$$

3. Tomando $u = ax + b$ luego

$$\begin{aligned} I = \int (ax+b)^n &= \int \frac{(u(x))^n}{a} u'(x) \\ &= \frac{1}{a} (\int u^n)_{u=ax+b} \\ &= \frac{(ax+b)^{n+1}}{a(n+1)} + C \end{aligned}$$

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4. Utilizando el teorema de cambio de variables

$$\begin{aligned} I = \int \frac{f'(x)}{f(x)} &= \left(\int \frac{1}{f} \right)_{f=f(x)} \\ &= \ln(f(x)) + C \end{aligned}$$

5. $\int \frac{1}{x^2+2x-3}$ notando que $x^2 + 2x - 3 = (x+3)(x-1)$ luego

$$\frac{1}{x^2 + 2x - 3} = \frac{A}{(x+3)} + \frac{B}{(x-1)}$$

Donde $A = -\frac{1}{4}$ y $B = \frac{1}{4}$ luego

$$\int \frac{1}{x^2 + 2x - 3} = \frac{1}{4}\left(-\int \frac{1}{(x+3)} + \int \frac{1}{(x-1)}\right) = \frac{1}{4}(-\ln(x+3) + \ln(x-1)) + C$$

Problema 1

1. $\int \frac{\cot x}{\ln(\operatorname{sen} x)} dx$

$I = \int \frac{\cot x}{\ln(\operatorname{sen} x)} dx$ Haciendo $u = \ln(\operatorname{sen} x) \;\Rightarrow\; du = \cot x\, dx$ con esto

$$I = \int \frac{1}{u} du = \ln u + C$$

retornando a las variables originales se obtiene

$$\int \frac{\cot x}{\ln(\operatorname{sen} x)} dx = \ln(\ln(\operatorname{sen} x))$$

Problema 2

Sea $I_n = \int \frac{\cos(nx)}{(\cos x)^n}$

1. Calcular $I_1,\ I_2$
2. Calcular $\int \frac{\operatorname{sen} x}{(\cos x)^{n+1}} dx$
3. Encontrar una relación de recurrencia para I_{n+1} en función de I_n

1. $I_1 = \int dx = x + C$, $I_2 = \int \frac{\cos(2x)}{(\cos x)^2}$ ademas $\cos 2x = \cos^2 x - \operatorname{sen}^2 x$ luego

$$\begin{aligned} I_2 &= \int \frac{\cos^2 x - \operatorname{sen}^2 x}{(\cos x)^2} \\ &= \int 1 - \tan^2 x \, dx \\ &= x - \int \tan^2 x \, dx \end{aligned}$$

para $\int \tan^2 x \, dx$ tomamos $u = \tan x \;\Rightarrow\; du = \sec^2 x \, dx = (1 + u^2) dx$

$$I_2 = x - \int \frac{u^2}{1 + u^2} dx = 2x - \tan x + C$$

2. $I = \int \frac{\operatorname{sen} x}{(\cos x)^{n+1}} dx$ sea $u = \cos x \;\Rightarrow\; -du = \operatorname{sen} x dx$

$$I = -\int \frac{1}{(u)^{n+1}} dx = \frac{1}{nu^n} + C$$

retornando a la variable original

$$I = \frac{1}{n(\cos x)^n} + C$$

3. $I_{n+1} = \int \frac{\cos((n+1)x)}{(\cos x)^{n+1}}$ pero $\cos((n+1)x) = \cos nx \cos x - \operatorname{sen} nx \operatorname{sen} x$

$$\begin{aligned} I_{n+1} &= \int \frac{\cos nx \cos x - \operatorname{sen} nx \operatorname{sen} x}{(\cos x)^{n+1}} \\ &= \int \frac{\cos nx \cos x}{(\cos x)^n \cos x} dx - \int \frac{\operatorname{sen} nx \operatorname{sen} x}{(\cos x)^{n+1}} dx \\ &= I_n - \underbrace{\int \operatorname{sen} nx \frac{\operatorname{sen} x}{(\cos x)^{n+1}} dx}_{J} \end{aligned}$$

J: por partes $u = \operatorname{sen} nx \;\Rightarrow\; du = n \cos nx \, dx$, $dv = \frac{\operatorname{sen} x}{(\cos x)^{n+1}} dx \;\Rightarrow\; v = \frac{1}{n(\cos x)^n} dx$ luego

$$\begin{aligned} I_{n+1} &= I_n - J \\ &= I_n - \left(\frac{\operatorname{sen} nx}{n(\cos x)^n} - \int \frac{\cos nx}{(\cos x)^n} dx \right) \\ &= 2I_n - \frac{\operatorname{sen} nx}{n(\cos x)^n} \end{aligned}$$