Economic Growth Theory

Benjamín Villena Roldán
CEA, Universidad de Chile

September 9, 2010

Overview of endogenous growth

- More general technologies such that Inada condition $\lim_{k\to\infty} f(k) = A > 0$. Higher factor substitution can overcome limited resources.
- Reproducible production factors, i.e. broader definition of capital, including human capital
- Capital accumulation spillover / Knowledge spillover (non-rival goods).
- Innovation in quantity of final / intermediate goods.
- Innovation in quality of goods (Schumpeterian models)

More general technologies (1)

- Elasticity of substitution is $\rho = -\frac{d\log(K/L)}{d\log(P_K/P_L)} = -\frac{f'(k)(f(k)-kf'(k))}{kf(k)f''(k)}$ (with perfectly competitive markets)
- Consider CES production function $Y = A\left(\alpha K^{\psi} + (1-\alpha)L^{\psi}\right)^{1/\psi}$ where $\rho = 1/(1-\psi)$.
- High substitution between K and L $(0 < \psi < 1 \rightarrow 1 < \rho < \infty)$ suffices to illustrate a violation to Inada conditions

$$y = f(k) = A\left(\alpha k^{\psi} + 1 - \alpha\right)^{1/\psi}$$

$$\partial y/\partial k = f'(k) = \alpha A\left(\alpha + (1-\alpha)k^{-\psi}\right)^{\frac{1-\psi}{\psi}}$$

More general technologies (2)

- Taking limits we get

$$\lim_{k\to\infty} \frac{f(k)}{k} = \lim_{k\to\infty} f'(k) = A\alpha^{1/\psi} > 0$$

- Hence, the production function asymptotically converges to a $\widetilde{A}k$ technology with $\widetilde{A} = A\alpha^{1/\psi}$ if $A > n + \delta$
- If the elasticity of substitution is low, i.e $\psi < 0 \to 0 < \rho < 1$ then the Inada condition holds

$$\lim_{k\to\infty} \frac{f(k)}{k} = \lim_{k\to\infty} f'(k) = 0$$

- Intuition: Exogenous growth of L prevents from sustained positive growth, but it can be easily replaced by K. For this reason, the marginal return of capital does not decrease to zero.

Endogenous growth: reproducible factors

- Key idea is reproducible production factors.
- Decreasing return to capital matters because labor and technology are growing at a fixed exogenous rate.
- But if there is an economic mechanism that reproduce factors, the limited availability of factors is avoided.
- Labor can be made reproducible if we allow for human capital accumulation.
- CES example: it also matters how easy non-reproducible factors can be substituted by reproducible factors.

One-sector endogenous growth with human capital

- Constant-return technology:
  $Y = F(K, H) = F(1, h)K = f(h)K$ with $h \equiv H/K$.
- Inada conditions, $f_h > 0$, $f_{hh} < 0$, $f(0) = 0$.
- Both H and K can be accumulated or consumed.
- Problem is represented recursively through

$$V(K, H) = \max_{K', H'} \{u(F(K, H) + (1 - \delta_K)K + (1 - \delta_H)H - K' - H') + \beta V(K', H')\}$$

- Two Euler equations (FOC+ Envelope)

$$u_c = \beta u_c'(F_K' + 1 - \delta_K)$$
$$u_c = \beta u_c'(F_H' + 1 - \delta_H)$$

Simple AK one-sector human growth (2)

- Net returns of both sectors equalize

$$
\begin{aligned}
F_K - \delta_K &= F_H - \delta_H \\
f(h) - f'(h)h - \delta_K &= f'(h) - \delta_H \\
\psi(h) \equiv f(h) - f'(h)(h+1) + \delta_H - \delta_K &= 0
\end{aligned}
$$

- Notice that $\psi'(h) > 0$, $\lim_{h\to 0} \psi(h) = -\infty$ and $\lim_{h\to\infty} \psi(h) = \infty$.
- Intermediate value thm $\Rightarrow$ unique $h^\star \in (0, \infty)$ solves equation.
- Call $f(h^\star) \equiv A$ so that $Y = f(h^\star)K = AK$
- Economy can sustain a positive growth rate forever.

Simple AK one-sector human growth (3)

- What's the dynamic behavior of this economy?
- If we use CRRA utility function, we get

$$c'/c = 1 + g_c = (\beta(1+R))^{1/\sigma} \quad \text{with } R = \frac{\partial y}{\partial k} - \delta = A - \delta$$

- Consumption grows at rate $g_c = (\beta(1 - \delta + A))^{1/\sigma} - 1$.
- Notice that growth rate does not vary with the level of k.
- This implies that there is no transition dynamics. All variables grow at the same rate forever.

Two-sector models

- Condition for endogenous growth: A "core" of capital goods using only reproducible factors direct or indirectly Rebelo (1991). A particular case is Lucas (1988) (based in Uzawa 1965).
- Suppose only K and C can be interchangeable consumed or accumulated.
- H can only be used for production. Let ϕ be share of K devoted to consumption-capital goods and θ be the share of H devoted to consumption-capital goods.
- Recursive formulation

$$V(K,H) = \max_{K',H',\phi,\theta} \{u(F(\phi K, \theta H) + (1-\delta_K)K - K') + \beta V(K',H')\}$$
$$\text{s.t.} \quad (1-\delta_H)H + G((1-\phi)K, (1-\theta)H) \geq H'$$

Two-sector models (2)

- First order conditions (Lagrange mult λ)

$$
\begin{aligned}
u: &\quad 0 = u_c F_1 K - \lambda G_1 K \\
v: &\quad 0 = u_c F_2 H - \lambda G_2 H \\
K': &\quad 0 = -u_c + \beta V_{K'} \\
H': &\quad 0 = \beta V_{H'} - \lambda = 0
\end{aligned}
$$

- Envelope conditions:

$$
\begin{aligned}
V_K &= u_c(F_1\phi + 1 - \delta_K) + \lambda G_1(1-\phi) \\
V_H &= u_c F_2 \theta + \lambda(1 - \delta_H + G_2(1-\theta))
\end{aligned}
$$

- First two conditions imply that marginal rate of transformation are equalized in both sectors

$$\lambda = u_c F_1 / G_1 = u_c F_2 / G_2$$

Two-sector models (3)

- λ is also the relative (shadow) price of H in terms of consumption.
- Substituting, the standard Euler equations are obtained

$$u_c = \beta u_c'(F_1' + 1 - \delta_K)$$
$$u_c = \beta u_c'(F_2' + (F_2'/G_2')(1 - \delta_H))$$

- These intertemporal and intratemporal conditions imply that the marginal return of both types of capital must be equal in both sectors.

$$F_1 - \delta_K = G_2 - \delta_H$$

Two-sector models (4)

- Add more structure: Suppose CRRA preferences and Cobb-Douglas technologies

$$F = A(\phi K)^{\alpha}(\theta H)^{1-\alpha} \quad G = A((1-\phi)K)^{\eta}((1-\theta)H)^{1-\eta}$$
$$u(c) = c^{1-\sigma}/(1-\sigma)$$

- Usual way to solve: guess-and-verify balanced growth path

$$1 + g_c = c'/c = [\beta(A(\theta H/\phi K)^{1-\alpha} + 1 - \delta_K)]^{1/\sigma}$$

- Constant consumption growth rate implies that $\theta H/\phi K = \tilde{h}$ is constant.

Two-sector models (5)

- Technologies and equality of returns in both sectors imply

$$\frac{1-\alpha}{\alpha}\frac{\phi}{1-\phi}=\frac{1-\eta}{\eta}\frac{\theta}{1-\theta}$$

- ϕ and θ are positively related.
- What happens if a corner solution occurs ($\phi = 1$)?

Two-sector models (6)

- Now take law-of-motion for H

$$\frac{H}{K}\frac{H'}{H} = (1-\delta_H)\frac{H}{K} + \frac{B((1-\phi)K)^{\eta}((1-\theta)H)^{1-\eta}}{(1-\phi)K}(1-\phi)$$

$$h(g_H+\delta_H) = Bh^{1-\eta}(1-\theta)\psi^{-\eta}$$

$$\text{with} \quad h \equiv \frac{\theta H}{\phi K} \quad \text{and} \quad \psi \equiv \frac{\alpha(1-\eta)}{(1-\alpha)\eta}$$

- Implies that θ is constant $\rightarrow$ ϕ is constant $\rightarrow$ H/K is constant $\rightarrow$ $g_K = g_H$.
- Now work out law-of-motion for K

$$\frac{C}{K} = A\phi h^{1-\alpha} - \delta_K - g_K$$

- Implies that C/K is constant. Hence $g_c = g_K = g_H = g$.

Two-sector models (7)

- Assume $\delta_K = \delta_H = \delta$ to get an analytical solution

$$F_1 = \alpha A(\theta H/\phi K)^{\alpha} = (1-\eta)B((1-\phi)K/(1-\theta)H)^{1-\eta} = G_2$$

- Then $h = \left(\frac{(1-\eta)B}{\alpha A \psi^{1-\eta}}\right)^{\frac{1}{1+\alpha-\eta}}$
- Using Euler equation we can see that

$$h = \left(\frac{(1+g)^{\sigma}}{A\alpha\beta} - \frac{1-\delta}{A\alpha}\right)^{\frac{1}{1-\alpha}}$$

Two-sector models (8)

- Doing some algebra the growth rate g depends on a geometric average of the TFP's in both sectors

$$(1+g)^{\sigma} = \beta\left(1-\delta+(\alpha A)^{\frac{2\alpha-\eta}{1+\alpha-\eta}}((1-\eta)B)^{\frac{1-\alpha}{1+\alpha-\eta}}\psi^{\frac{-(1-\eta)(1-\alpha)}{1+\alpha-\eta}}\right)$$

- Changing marginal return of capital (via taxes) generate permanent effects on growth rates, i.e. widening GDP gaps across countries with different policies.

Two-sector models (9)

- What is needed to have endogenous growth, i.e. $g > 0$ after all?
- Mulligan and Sala-i-Martin (1993) set this "generalized model"

$$Y = C + K' - (1-\delta_K)K = A(\phi K)^{\alpha_1}(\theta H)^{\alpha_2}$$
$$H' = (1-\delta_H)H + B((1-\phi)K)^{\eta_1}((1-\theta)H)^{\eta_2}$$

- Assume balanced growth and constant ϕ, θ. Same arguments as before.
- Since $Y/K = C/K + K'/K - (1-\delta_K)$ and $g_K = K'/K$ is constant, $g_Y = g_K = g_C$.
- Why? With constant growth rate gap, the budget constraint will be violated.
- Also if $g_K > g_C$ limiting marginal product of capital is incompatible with transversality condition.

Two-sector models (10)

- Let's do some algebra and remembering that g_H is constant

$$Y'/Y = g_Y = g_K = (K'/K)^{\alpha_1}(H'/H)^{\alpha_2}$$
$$\frac{H''/H' - (1-\delta_H)}{H'/H - (1-\delta_H)} = 1 = (K'/K)^{\eta_1}(H'/H)^{\eta_2}$$

- We obtain following system with $\hat{g}_x = \log g_x$

$$(\alpha_1 - 1)\hat{g}_K + \alpha_2\hat{g}_H = 0$$
$$\eta_1\hat{g}_K + (\eta_2 - 1)\hat{g}_H = 0$$

- Trivial solution is $\hat{g}_K = \hat{g}_H = 0$ no endogenous growth, i.e. steady state solution

Two-sector models (11)

- Solution with $\hat{g}_K \neq 0$ and $\hat{g}_H \neq 0$ if $\alpha_2\eta_1 = (1-\alpha_1)(1-\eta_2)$
- Condition is satisfied if there are constant returns in both sectors, i.e. $\alpha_1 = 1-\alpha_2$ and $\eta_1 = 1-\eta_2$
- It also holds if $\eta_1 = 0$ and $\eta_2 = 1$ for any α_1, α_2 (Uzawa-Lucas model). Linear accumulation technology of human capital.
- Providing $\alpha_1 \neq 1$, we know conclude that $\hat{g}_K = \frac{\alpha_2}{1-\alpha_1}\hat{g}_H$
- Hence, $\hat{g}_K \lesseqgtr \hat{g}_H$ as $\alpha_1 + \alpha_2 \lesseqgtr 1$.
- Only $\alpha_1 + \alpha_2 = 1$ is compatible with constant long-run K/H ratio.

Lucas, R. E. (1988).
On the Mechanics of Economic Development.
Journal of Monetary Economics 22(1), 3 – 42.

Mulligan, C. B. and X. Sala-i-Martin (1993).
Transitional Dynamics in Two-Sector Models of Endogenous Growth.
The Quarterly Journal of Economics 108(3), 739–773.

Rebelo, S. (1991).
Long-Run Policy Analysis and Long-Run Growth.
The Journal of Political Economy 99(3), 500–521.