Economic Growth Theory

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August 23, 2010

Neoclassical growth with Dynamic Programming

- Also known as Ramsey or Cass-Koopmans model.
- Standard assumptions: in this economy households,
  - Are risk averse ($u(\cdot)$ is concave) and live forever.
  - Own capital stock.
  - Supply one unit of labor inelastically.
  - Discount future at rate β.
  - Freely transform 1 unit of c into an unit of k.
- Households' objective is to maximize lifetime utility subject to technological constraints.

$$\max_{\{c_t,k_{t+1}\}_{t=0}^{\infty}} \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) \right\}$$

$$\text{subject to} \quad c_t + k_{t+1} \leq f(k_t) + (1-\delta)k_t$$

Neoclassical growth (2)

- We can write the problem as

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \left\{ \sum_{t=0}^{\infty} \beta^t u(f(k_t) + (1-\delta)k_t - k_{t+1}) \right\}$$

- Household decide how much to consume and how much capital to accumulate by choosing k_{t+1}
- k_t fully determines the amount of resources available in t (k is a state variable)
- Given state variable k_τ, we can completely describe a feasible plan $\{k_{t+1}\}_{t=\tau+1}^{\infty}$ of choice variables.
- State variables fully characterize the value of the objective function.
- Control variables are current period's choices to maximize objective.

Neoclassical growth (3)

- Find the optimal sequence that maximizes lifetime utility $\{k_{t+1}\}_{t=0}^{\infty}$
- The problem can be rewritten as

$$\begin{aligned} V(k_0) = \max_{k_1} \Bigg\{ & u(f(k_0) + (1-\delta)k_0 - k_1) \\ & + \beta \max_{\{k_{t+1}\}_{t=1}^{\infty}} \left\{ \sum_{t=0}^{\infty} \beta^t u(f(k_t) + (1-\delta)k_t - k_{t+1}) \right\} \Bigg\} \\ V(k_0) = \max_{k_1} & \{u(f(k_0) + (1-\delta)k_0 - k_1) + \beta V(k_1)\} \end{aligned}$$

- If there exists the function $V(k)$, we can use standard optimization techniques.

Neoclassical growth (4)

- Recursive approach allows to solve a functional equation: Bellman equation.

$$V(k) = \max_{k'} \{u(f(k) + (1-\delta)k - k') + \beta V(k')\}$$

- Does $V(k)$ exist? What are the properties of the Value Function?
- Contraction Mapping Theorem also known as Banach Fixed Point Theorem.
- Once existence is determined, it's a standard maximization problem.

Neoclassical growth (5)

- First-order condition is

$$-u_c(c) + \beta V'(k') = 0$$

- Envelope theorem $V'(k) = u_c(c)(f_k(k) + 1 - \delta)$. Why?
- Remember that $k' = g(k)$, hence

$$\begin{aligned} V'(k) = dV(k)/dk &= u_c(c)(f_k + 1 - \delta - g'(k)) + \beta V'(g(k))g'(k) \\ &= u_c(c)(f_k + 1 - \delta) + g'(k)\underbrace{(-u_c + \beta V'(k'))}_{=0 \quad \text{FOC}} \end{aligned}$$

Euler Equation

- Replacing previous result yields the Euler equation

$$u_c(c) = \beta u_c(c')(f_k(k') + 1 - \delta)$$

- Usual Constant Relative Risk Aversion (CRRA) preferences: $u(c) = c^{1-\sigma}/(1-\sigma)$. If $\sigma = 1$, $u(c) = \log c$. Why? $-u''(c)c/u'(c) = \sigma$.
- CRRA preferences are compatible with balanced growth path for constant return of capital

$$g_c = c'/c = (\beta(f_k(k') + 1 - \delta))^{1/\sigma}$$

Transversality Condition (1)

- Remember that FOC are necessary but not sufficient for a maximum.
- Suppose that we have a finite horizon economy.

$$\max_{\{k_{t+1}\}_{t=0}^{T}} \left\{ \sum_{t=0}^{T} \beta^t u(f(k_t) + (1-\delta)k_t - k_{t+1}) \right\}$$
$$\text{s.t.} \quad k_0 > 0, k_{t+1} \geq 0$$

- This is a finite-dimensional optimization problem $\rightarrow$ directly solved using Kuhn-Tucker theorem.
- Suppose that solution $k_{t+1}^{\star} > 0$ so that no complementary slackness conditions are required.

Transversality Condition (2)

- Since $c_t = f(k_t) + (1-\delta)k_t - k_{t+1}$, FOC are

$$0 = -u'(c_0) + \beta u'(c_1)(f'(k_1) + 1 - \delta)$$
$$\dots\dots\dots$$
$$0 = -u'(c_{T-1}) + \beta u'(c_T)(f'(k_T) + 1 - \delta)$$

- However, for the choice of k_{T+1}, the complementary slackness condition is important.

$$\beta^T u'(c_T) k_{T+1} = 0$$

- Either no capital is left at the end of the world or the shadow price of that capital is zero.
- Since k can be transformed into c and utility increases in c, it's clear that $k_{T+1} = 0$.

Transversality Condition (3)

- What happens if $T \to \infty$? Heuristic extension suggests

$$\mathcal{L} = \lim_{T\to\infty} \beta^T u'(c_T) k_{T+1} = 0$$

- If $\mathcal{L} > 0$ a higher lifetime utility is achieved by consuming the excessive capital accumulation.
- If households hold assets instead of capital and the transversality condition is violated such that $\mathcal{L} = \lim_{T\to\infty} \beta^T u'(c_T) a_{T+1} < 0$, households roll over the debt forever.

Transversality Condition (4)

- Ponzi game: using new debt to pay the interests of previous debt. The principal is never paid back, so that consumption can grow without bound. $\mathcal{L} = 0$ is also called no-Ponzi condition.
- Bottom line: Euler equation + Transversality are necessary and sufficient conditions for optimal growth path $\{k^\star_{t+1}\}_{t=0}^{\infty}$ for a given $k_0 > 0$.

Steady state and Balanced Growth (1)

- As in most cases "guess-and-verify" a balanced growth path: all endogenous variables grow at (possible different) constant rate.
- Capital law-of-motion

$$c/k + g_k = 1 - \delta + f(k)/k$$

- c and k must grow at same rate; otherwise, budget constraint is violated or growth rate isn't constant.
- $f(k)/k$ is strictly decreasing and f satisfies Inada conditions, so at some point crosses $(c/k)^{\star} + g_k - 1 + \delta$.

Steady state and Balanced Growth (2)

- It follows there is unique $k^\star$ and that $g_k = 1$ because k is constant in the long-run, i.e. there is a steady-state.
- Steady state $c' = c = c^\star$ and $k' = k = k^\star$

$$\beta(f_k(k^\star) + 1 - \delta) = 1 \quad \Rightarrow \quad k^\star = f_k^{-1}(1/\beta - 1 + \delta)$$

- Moreover, the consumption is $c(k^\star) = f(k^\star) - \delta k^\star$.

Competitive Equilibrium (1)

- Result can be decentralized in several ways
- No distortions, no frictions in the economy so that First Welfare Theorem applies: the competitive equilibrium is Pareto Optimal.
- Alternative setup: Households accumulate assets and finance firms which own the capital.

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) \right\}$$

$$\text{s.t.} \quad c_t + a_{t+1} \leq w(K,L)l + (1 + r(K,L))a_t$$

$$a_0 \quad \text{given}$$

$$l_t \leq 1$$

Competitive Equilibrium (2)

- Firms maximize profits $\pi = F(K,L) - w(K,L)L - R(K,L)K$ by choosing how much labor to hire and how much capital to rent
- In equilibrium firms are price-takers and hire K_t and L_t up to the point $F_K(K_t, L_t) = R(K_t, L_t)$ and $F_L(K_t, L_t) = w(K_t, L_t)$.
- Households face interest rate $r(K_t, L_t) = R(K_t, L_t) - \delta$.
- In equilibrium $a_t = K_t$ and $l_t = L_t = 1$ such that markets clear at competitive prices.

Competitive Equilibrium (3)

- Formally a competitive equilibrium is a set of paths $\{Y_t, C_t, K_{t+1}\}_{t=0}^{\infty}$ and sequences of prices $\{r_t, R_t, w_t\}_{t=0}^{\infty}$ such that
  - Given an initial a_0 and prices $\{r_t, w_t\}_{t=0}^{\infty}$, households maximize their lifetime utility.
  - Given prices $\{R_t, w_t\}_{t=0}^{\infty}$, firms maximize their profits.
  - Markets clear, that is $a_t = K_t$ and $l_t = L_t = 1$.
- Same allocation as in previous setup where households own capital and produce directly.

Studying dynamics around steady state (1)

- Euler is also second-order nonlinear dynamic equation in k

$$u_c(f(k)+(1-\delta)k-k') = \beta u_c(f(k')+(1-\delta)k'-k'')(f_k(k')+1-\delta)$$

- To study local dynamics we log-linearize around steady state

$$0 = u_{cc}dc - \beta u_{cc}\underbrace{(f_k+1-\delta)}_{=\beta^{-1}\text{ in SS}}dc' - \beta u_c f_{kk} dk'$$

- It is also true that

$$dc = (f_k+1-\delta)dk - dk' = \beta^{-1}dk - dk'$$

- Hence, if $\widehat{k} = dk/k^\star$ we have that

$$0 = \beta^{-1}\widehat{k} - \left(1+\beta^{-1}+\beta\frac{u_c}{u_{cc}}f_{kk}\right)\widehat{k}' + \widehat{k}''$$

- How to solve this dynamic equation?

Studying dynamics around steady state (2)

- "Guess-and-verify" + Undetermined coefficients method.
- Conjecture: Solution has the form $\widehat{k}' = \phi\widehat{k}$.
- Replacing the conjecture and denoting $a = (1 + \beta^{-1} + \beta\chi)$ and $\chi = \frac{u_c}{u_{cc}} f_{kk} > 0$ we obtain

$$\beta^{-1}\widehat{k} - a\phi\widehat{k} + \phi\widehat{k}' = 0 \quad (a - \beta^{-1}/\phi)\widehat{k} = \widehat{k}'$$

- Conjecture was right!
- Solution: quadratic equation $\phi^2 - a\phi + \beta^{-1} = 0$. Two roots:

$$\phi = \frac{a \pm \sqrt{a^2 - 4\beta^{-1}}}{2}$$

$$\phi = \frac{(1 + \beta^{-1} + \beta\chi) \pm \sqrt{(1 + \beta^{-1} + \beta\chi)^2 - 4\beta^{-1}}}{2}$$

Studying dynamics around steady state (3)

- If $\chi = 0$, then the solutions are $\phi(-) = 1$ and $\phi(+) = \beta^{-1}$.
- Now, let's determine how χ affects ϕ

$$\frac{\partial \phi}{\partial \chi} = \frac{1}{2}\left(\beta \pm \frac{(1+\beta^{-1}+\beta\chi)\beta}{\sqrt{1+\beta^{-1}+\beta\chi - 4\beta^{-1}}}\right)$$
$$= \frac{\beta}{2}\left(1 \pm \frac{1}{\sqrt{1-\frac{4\beta^{-1}}{(1+\beta^{-1}+\beta\chi)^2}}}\right)$$

- Second expression in parenthesis is greater than 1. Hence $\phi(-)$ decreases in χ while $\phi(+)$ increases in χ.
- Then $\phi(-) < 1 \rightarrow$ stable solution: system goes back to steady state.
- Then $\phi(+) > \beta^{-1} > 1 \rightarrow$ unstable solution: system diverges from steady state.

Optimal Control Approach

- Alternative approach to Dynamic Programming. It conceptually yields the same result.
- Key difference: time is continuous. Instead of determining an optimal sequence, we look for an optimal function.
- Households maximize

$$\int_0^\infty e^{-\rho t} u(c(t))dt$$

$$\begin{aligned} \text{s.t.} \quad & (a)\dot{k} = f(k) - \delta k - c \\ & (b)k(0) = k_0 > 0 \\ & (c)\lim_{t \to 0} k(t)e^{-\bar{r}(t)t} = 0 \end{aligned}$$

- Heuristic derivation and Cookbook recipe in Barro and Sala-i-Martin (2004) Appendix. More complete treatment in Acemoglu (2009) chapter 7.

Acemoglu, D. (2009).
Introduction to Modern Economic Growth.
Princeton University Press.

Barro, R. J. and X. Sala-i-Martin (2004).
Economic Growth (Second ed.).
The MIT Press.