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Abstract

Ratios that indicate the statistical significance of a fund's alpha typically appraise its performance. A growing literature suggests that even in the absence of any ability to predict returns, holding options positions on the benchmark assets or trading frequently can significantly enhance performance ratios. This paper derives the performance-maximizing strategy—a variant of buy-write—and the least upper bound on such performance enhancement, thereby showing that if common equity indexes are used as benchmarks, the potential performance enhancement from trading frequently is usually negligible. The enhancement from holding options can be substantial if the implied volatilities of the options are higher than the volatilities of the benchmark returns.

Key words: alpha, hedge funds, mutual funds, portfolio management, options

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1 Introduction

Since Jensen's (1968) seminal work, there is a growing literature[1] that questions whether a positive alpha does in fact imply a manager's ability to select assets by predicting returns. A number of researchers[2] answer the question by pointing out, through simulation and examples, that by investing in options, managers can generate a positive alpha relative to a benchmark even if they cannot predict returns. However, up to this point the answers to the question have been *ad hoc*, leaving the literature unclear about the significance of alpha that can be achieved by trading frequently or holding derivative securities.

This paper delivers the theoretical answer to the question by deriving the explicit formulas for the trading strategy that maximizes alpha and its significance. The performance-maximizing strategy is shown to be a variant of a buy-write strategy, which can be implemented by taking long positions in the benchmark assets and writing options on them. The manager's ability to generate superior performance from trading frequently or holding derivatives is shown to be by and large negligible under the Black-Scholes model. It is easier for him to seem successful even in the absence of superior information when the implied volatility of derivative securities is higher than the volatility of the underlying benchmark securities. These results provide the theoretical implications on alpha of allowing a manager to trade while being evaluated with respect to a buy-and-hold portfolio of the given benchmark.

The starting point is the consideration of a set of risky securities a fund manager can trade. The manager aims to, and often claims to deliver superior performance. Clients, potential clients and academics evaluate such claims based on a set of benchmark assets, which are available to the public, including the managers' clients and potential clients. The challenge is to evaluate the fund's return by asking: Could it be obtained from a portfolio of the benchmark assets? If not, what is the increment of the evaluated return relative to the return available in the benchmark space? How likely is the increment strictly positive? The increment is labeled as alpha. This measure of performance is widely implemented in practice, and financial information services report alpha for securities and funds. Some hedge funds have even included the word "alpha" as part of their names.

If the manager delivers a series of returns which are on average higher than those obtainable from the benchmark assets, the evaluator may infer that the manager can outperform the benchmark. The attribution of such out-performance to superior information, however, may be premature because there are at least four sources for the manager's access to a

[1] The literature can be traced back to Dybvig and Ingersoll (1982), Grinblatt and Titman (1989) and Glosten and Jagannathan (1994).

[2] See Lo (2001), Hill et al. (2006), Goetzmann et al. (2007).

larger space of payoffs than the space of the benchmarks. First, the manager can predict the returns of the benchmark assets and can choose portfolio weights to reflect his predictions. Second, the manager may trade securities with payoffs outside the benchmark space. Third, the manager may trade derivatives of the benchmark assets. Fourth, the manager may trade the benchmark assets more frequently than the evaluator observes the returns.

The last two sources only give the appearance of superior performance because they do not depend on the manager's access to superior information. The two sources are in fact equivalent if derivative securities are redundant assets and their payoffs can be replicated by judiciously trading the underlying assets. Black and Scholes (1973) and Merton (1973) point out the equivalence between the payoff on derivatives and on such rule-based trading. Following the literature, this paper refers to rule-based trading designed to replicate a derivative payoff as delta trading.

The specific building blocks of the performance evaluation mechanism studied here are familiar. The evaluator decomposes the fund's return into two parts—the return's linear projection on the benchmarks and the incremental return orthogonal to the benchmarks. The expectation of the incremental return is alpha, representing the portion of the expected return that is not attainable by passive investment strategies in the benchmarks. The tracking error, which is the standard deviation of the incremental return, measures the uncertainty of alpha.

The resulting ratio of alpha to the tracking error is typically referred to as the appraisal ratio. A high appraisal ratio is interpreted as superior performance because it indicates a large probability of a positive return after subtracting financing costs and neutralizing the risks associated with the benchmarks. Maximizing the appraisal ratio is necessary for maximizing the Sharpe ratio, a widely used measure of fund performance. For a hedge fund, the appraisal ratio of the fund is by itself the Sharpe ratio of the hedged position that neutralizes the benchmark risk. More importantly, the appraisal ratio is the asymptotic t statistic of the estimated alpha.

Sharpe ratios and appraisal ratios are important in practice. (See, e.g., Treyner, 2008) Nonetheless, within a fund class, e.g., large cap long-only US equity, it is common to compare fund performance according to the funds' alpha rather than consider their appraisal ratios. This practice is reasonable when the leverage is similar across the compared funds. (e.g., zero leverage in the large cap long-only US equity.) This practice is less reasonable when comparing funds with different leverage or different sets of underlying assets (and therefore different benchmarks and tracking errors). Then attention naturally turns to the Sharpe ratio.

This paper derives the best appraisal ratio (and thus the best asymptotic t statistic of alpha) a manager can obtain. Section 2 sets up the general problem and describes the benchmark space used by the evaluator and the payoff space accessible to the manager. The fundamental results are in Theorem 1 which is in Subsection 2.1: as the number of payoff observations becomes large, (i) the maximal appraisal ratio is the safe return times the norm of the difference of the pricing kernels of the two spaces, and (ii) the maximal appraisal ratio is achieved by a payoff which can be described as a function of the pricing kernels. Subsection 2.2 discusses how constraints on the tracking error or on leverage transform an upper bound on the appraisal ratio into an upper bound on alpha. Subsection 2.3 offers two new results, Theorems 2 and 3, which show that the performance of the benchmark-switching schemes suggested in Goetzmann et al. (2007) deteriorates as the observation period increase. Moreover, this subsection shows that for reasonable parameter configurations these benchmark-switching schemes deliver on average sizable alphas - around 2% – but these alphas are too noisy to be statistically significant.

A novelty of the paper is in its examination of a complete market with multiple benchmark assets. In Section 3, Theorem 4 gives a simple formula of the optimal appraisal ratio in terms of the moments of the benchmark assets. The subsequent discussion shows that trading derivatives on the benchmark assets will deliver only a small improvement of the Sharpe ratio and will take thousands of monthly observations to produce a statistically significant alpha. The improvement can however be significant when less frequent observations or a larger benchmark space are considered (e.g., by including certain additional factors).

Another novelty of the paper is in its examination of the optimal trading strategies. Section 4 establishes that in the case of a single benchmark asset, the strategy that maximizes the appraisal ratio consists of holding the benchmark and writing out-of-the money options on it. Under the assumption that the Black-Scholes model governs option pricing, the best strategy still delivers low values of the appraisal ratio if the parameters are estimated from the equity indices. These low values of the appraisal ratio are in contrast with the practical experience in which holding the S&P 500 and writing at-the-money call options on it has delivered economically and statistically significant alphas. Theorem 5 offers a reconciliation of the contradiction by showing that the optimal appraisal ratio is high if options are not redundant assets and are priced at an implied volatility that is higher than the volatilities of the benchmark assets.

Section 5 concludes with a discussion of the past and future research in this area.

2 Performance Maximization

In practice, a fund manager who is evaluated by his fund's performance relative to an index can hold the index and write options on it. (The S&P 500, Nasdaq 100 and Russell 2000 indices are examples of indices on which options can easily be written.) The intuition for the consequent payoff is illustrated in Figure 1, which is reprinted from Wang and Zhang (2003). If the fund is fully invested in the index, the fund return (the dashed line) is a linear function of the market return with zero intercept. If the manager writes call options on the index and invests the proceeds in the safe asset, the fund return (the dotted-dashed line) is a nonlinear function of the market return. If the fund consists of a long position in the index and a short position in the options, the fund return (the thick line) has a nonzero alpha in the regression of the fund's excess return on the excess return of the index (the dotted line).

This demonstration of the possibility of generating alpha is neither an algorithm which generates the highest possible alpha nor a proof that the highest appraisal ratio can be generated in this way. The first subsection introduces the paper's basic framework, which to a large extent Theorem 1 captures. Theorem 1 is general and to study special cases in later sections, further assumptions are made. The second subsection applies Theorem 1 to trading strategies that have been studied in the literature and shows them to be less attractive than it had been argued.

2.1 The optimization problem and its solution

The excess return r_x on an actively managed fund is evaluated against a vector of excess returns $r_m = (r_1, \cdots, r_k)'$ on k benchmark assets. The period during which the fund is evaluated is from time 0 to time T. Let Δt, $2\Delta t$, $\cdots$, $n\Delta t$, where $n\Delta t = T$, be the equally-spaced time grid on which the evaluator observes the returns. Denote by r_{xi} and r_{mi} the observed excess returns on the fund and the benchmarks, respectively, over the time interval from $(i-1)\Delta t$ to $i\Delta t$. The regression of the excess fund returns on the excess benchmark returns is

$$r_{xi} = \alpha + r'_{mi}\beta + \epsilon_i \ , \tag{1}$$

where the intercept α and the vector of slope coefficients β satisfy

$$\alpha = E[r_x - r'_m\beta] \tag{2}$$

$$\beta = (\text{var}(r_m))^{-1}\text{cov}(r_m, r_x) \ . \tag{3}$$

After hedging out the risk associated with the benchmarks by adding a short position of β in the benchmark assets, the return of the hedged position is, $r_x - r'_m\beta$, its expectation is α, and its tracking error is $\sqrt{\text{var}(r_x - r'_m\beta)}$. The appraisal ratio of the hedged position is its Sharpe ratio: the ratio of alpha to its tracking error. It is

$$\text{APR} = \frac{\alpha}{\sqrt{\text{var}(r_x - r'_m\beta)}} \ . \tag{4}$$

A high appraisal ratio indicates high profitability for the hedged position. The appraisal ratio is also the asymptotic t statistic of alpha—the limit of the t statistic multiplied by the square-root of the number of observations in the ordinary least-square (OLS) regression (1). Thus, a high appraisal ratio is associated with a high t statistic of alpha in that regression.

The manager of the fund who seeks to maximize the appraisal ratio should have a trading strategy such that the fund return r_x solves the following maximization problem:

$$\max_{x} \ \frac{E[r_x - r_m' \beta]}{\sqrt{\text{var}(r_x - r_m' \beta)}} \, . \tag{5}$$

The maximization problem captures the tradeoff the portfolio manager faces: to maximize the appraisal ratio, the return should have not only a high alpha but also a low tracking error (i.e., a low standard deviation of the residual return).

Before presenting the solution to the maximization problem, it is necessary to describe the payoff space and the pricing function. The space of payoffs attainable by the manager in a given period by trading in the securities available to him, denoted by X_a, is the *attainable space*. The payoffs are assumed to have finite second moments, that is $X_a \subset L^2(P, \Omega)$, where Ω is a sample space and P is a probability measure, and $L^2(P, \Omega)$ is the set of all measurable functions with second moments. For any $x \in L^2(P, \Omega)$, the square-root of its second moment is the norm of x, denoted by $\|x\| = E[x^2]^{1/2}$. The assumption of finite second moment is a minimal requirement to ensure that the sample estimates of linear regressions converge to their population counterparts.

The assumptions on the attainable space are standard and flexible, allowing the market to be incomplete because the space X_a is allowed to be a strict subset of $L^2(P, \Omega)$. The dimension of X_a may be infinite. This is important because a payoff space that contains options with infinitely many strike prices and maturities has an infinite dimension. The only restriction is the linearity of the payoff space, which excludes the presence of constraints or frictions that a fund manager may face in the market.

The linear space of payoffs spanned by the benchmark assets, denoted by X_b, is the *benchmark space*. The number of benchmark assets is typically small and the dimension of X_b is assumed to be $k+1$. Let $\{x_j\}_{j=0,\cdots,k}$ be the payoffs of the $k+1$ independent assets that span X_b. For simplicity and practical applications, the first benchmark payoff x_0 is assumed to be the constant payoff of a safe asset. If the payoffs in X_b do not exhaust the set of payoffs attainable by the manager, then X_b is a strict subset of X_a. A fund has abnormal return relative to the benchmarks if and only if its return or payoff falls outside the benchmark space.

Let $v : X_a \mapsto \Re$ be the pricing function, where $\Re$ is the set of real numbers. Assume that the law of one price holds, and thus v is linear. A stochastic discount factor for X_a is a random variable $m \in L^2(P, \Omega)$ such that $v(x) = E[xm]$ for all $x \in X_a$. Denote the set of all stochastic discount factors for X_a by M_a. By the Riesz representation theorem, there exists some $m_a \in X_a$ such that $v(x) = E[xm_a]$ for all $x \in X_a$, i.e., $m_a \in X_a \cap M_a$. It follows that m_a has the smallest norm among all the stochastic discount factors in M_a (see chapter V of Riesz and Sz.-Nagy (1955)). Since the price function also applies to the set of benchmark assets, the set of the stochastic discount factors for X_b is $M_b = \{m \in L^2(P, \Omega) : v(x) = E[mx] \text{ for all } x \in X_b\}$, and there exists a smallest-norm discount factor $m_b \in X_b \cap M_b$.

A trading strategy of the fund corresponds to a payoff $x \in X_a$. To obtain excess returns, assume that the payoff of the safe asset is $x_0 = 1$ and its price is $v(1) > 0$. The return on the safe asset is $R_0 = 1/v(1)$. The excess return on the fund is $r_x = x - v(x)R_0$, and the excess return on the jth benchmark is $r_j = x_j - v(x_j)R_0$ (for $j = 1, \cdots, k$). Let $r_m = (r_1, \cdots, r_k)'$. Based on the observations of (r_x, r'_m) over n time intervals of equal length, the sample estimates of alpha (denoted by $\hat{\alpha}_n$) and appraisal ratio (denoted by $\widehat{\text{APR}}_n$) are obtained in the OLS regression (1). The sample estimate of the appraisal ratio is in fact the t statistic of $\hat{\alpha}_n$ multiplied by $\sqrt{n}$. As $n \to \infty$, the sample estimate converges to their population counterparts, i.e., $\lim \hat{\alpha}_n = \alpha$ and $\lim \widehat{\text{APR}}_n = \text{APR}$, where α and APR are defined in equations (2) and (4), respectively. The alpha and appraisal ratio depend on the fund's strategy x, and therefore are denoted by $\alpha(x)$ and $\text{APR}(x)$.

The optimization problem is to find a strategy $x \in X_a$ that maximizes $\text{APR}(x)$. The maximal appraisal ratio is denoted by

$$\text{APR}_{\max} = \max\{\text{APR}(x) : x \in X_a\}\,. \tag{6}$$

The solution to the optimization problem (6) is similar in spirit to the construction of the mean-variance frontier in Hansen and Richard (1987) and Huberman and Kandel (1987), starting with the observation that any payoff can be decomposed into three components with uncorrelated payoffs. The first component accounts for beta and is alpha-neutral; the second one accounts for alpha and is beta-neutral; the third component is alpha- and beta-neutral and only adds variance. Therefore the best tradeoff between alpha and the variance of the regression residual is achieved for those returns in which only the first two components are present. The solution to (6) is characterized in the following theorem.

Theorem 1 *The alpha of any payoff $x \in X_a$ is:*

$$\alpha(x) = R_0 E[r_x(m_b - m_a)]\,. \tag{7}$$

The maximal appraisal ratio over all the payoffs in X_a is

$$\text{APR}_{\max} = R_0 \|m_b - m_a\|\ , \tag{8}$$

and the maximum is achieved for any payoff x of the form:

$$x = z + \theta(m_b - m_a) \tag{9}$$

for some $z \in X_b$ and $\theta > 0$.

The proofs of the theorems are relegated to the Appendix.

Theorem 1 is closely linked to earlier analyses of payoff spaces. This should not surprise a reader familiar with the interpretation of asset payoffs as elements of Hilbert spaces. Numerous papers have studied asset payoffs in that context. The closest earlier pieces appear to be Chamberlain and Rothschild (1983), Hansen and Richard (1987), and Hansen and Jagannathan (1991, 1997).

Hansen and Jagannathan (1997) apply the properties of L^2 norms to the comparison of asset-pricing models. The right-hand side of equation (8) is the product of R_0 and the L^2 norm of $m_b - m_a$; the latter term can also be interpreted as the the Hansen and Jagannathan's (HJ) distance of m_b from the set of discount factors that price all payoffs in X_a. The HJ distance is

$$\delta = \min\left\{\|m_b - m\| : m \in M_a\right\}. \tag{10}$$

Any discount factor $m \in M_a$ can be written as $m = m_a + (m - m_a)$ with $E[(m - m_a)x] = 0$ for all $x \in X_a$. It follows from $m_b - m_a \in X_a$ that

$$\|m_b - m\|^2 = \|m_b - m_a\|^2 + \|m - m_a\|^2 . \tag{11}$$

The minimum of $\|m_b - m\|$ over $m \in M_a$ is achieved when $\|m - m_a\| = 0$, and thus $\text{APR}_{\max} = R_0\delta$. In this interpretation, m_b is understood as a given asset-pricing model, and δ measures the misspecification of the model.

The maximal appraisal ratio is related to the variance bounds derived by Hansen and Jagannathan (1991). With R_0 denoting the safe rate, equation (8) reduces to:

$$\text{APR}_{\max} = R_0\sqrt{\text{var}(m_a) - \text{var}(m_b)} \tag{12}$$

because $E[m_b m_a] = \|m_b\|^2$ and $E[m_b] = E[m_a] = 1/R_0$. According to Hansen and Jagannathan, $\text{var}(m_a)$ is the greatest lower bound of the variance of the stochastic discount factors in M_a, and the same statement holds for $\text{var}(m_b)$ and M_b. It then follows from equation (12) that the square root of the difference between the two variance bounds gives the maximal squared appraisal ratio discounted by R_0.

The maximal appraisal ratio is also related to the Sharpe ratios of the payoff spaces X_b and X_a, which are

$$\text{SHP}_i = \max\{E[r_x]/\text{var}(r_x)^{1/2} : x \in X_i\} \quad \text{for } i = a, b. \tag{13}$$

According to Hansen and Jagannathan (1991), the Sharpe ratio of a payoff space is the standard deviation of the smallest-norm discount factor scaled by the safe return, i.e.,

$$\mathrm{SHP}_i = R_0\sqrt{\mathrm{var}(m_i)} \quad \text{for } i = a, b \;, \tag{14}$$

which imply

$$\mathrm{APR}_{\max} = \sqrt{\mathrm{SHP}_a^2 - \mathrm{SHP}_b^2} \;. \tag{15}$$

Therefore, as one enlarges the payoff space from X_b to X_a, the improvement in the squared Sharpe ratio is the squared maximal appraisal ratio $\mathrm{APR}_{\max}$. The relation between the appraisal ratio and the Sharpe has appeared in the literature under various specialized assumptions. Goeztmann et al. (2007) derive this relation for the case where security prices are discrete (see equation 7 in that paper) and markets are complete. In the case that X_a consists of payoffs of the benchmarks and a set of additional assets, $\mathrm{APR}_{\max} = 0$ if and only if the efficient frontier of X_a is spanned by X_b, as discussed by Huberman and Kandel (1987). The simplest version of equation (15), in which X_b consists of only the market return and X_a includes an additional stock return, can be traced back to Treynor and Black (1973).

In summary, Theorem 1 covers or implies multiple existing results in the literature. Besides its generality, the theorem allows novel applications, which will appear in Sections 3 and 4, in the analysis of maximal performance ratio and derivative trading.

2.2 Maximizing alpha

Theorem 1 maximizes the appraisal ratio, which is linked to the significance of alpha, but does not maximize alpha itself. Indeed, because alpha scales with leverage, if there is a zero price payoff with nonzero alpha, then there are payoffs with arbitrarily large alphas. In practice, such large alphas may not be attainable at least for two reasons: limits on risk and collateral requirements, which both imply constraints on leverage.

First, money managers may not exceed a certain level of risk. A typical risk management mandate entails that the tracking error, defined as the standard deviation of the hedged payoff, be lower than a certain upper bound TE:

$$\sqrt{\mathrm{var}(r_x - r_m'\beta)} \leq \mathrm{TE}$$

Thus, a manager with such a mandate will choose a payoff of the form:

$$x = z + \mathrm{TE}\frac{m_b - m_a}{\sqrt{\mathrm{var}(m_b) - \mathrm{var}(m_a)}}$$

where z is an arbitrary payoff in the benchmark space. Such a choice scales a payoff with maximum appraisal ratio to the maximum tracking error TE allowed, obtaining a maximum alpha equal to the maximum appraisal ratio, times the maximum tracking error:

$$\alpha_{\max} = \text{APR}_{\max}\text{TE} = R_0\sqrt{\text{var}(m_b) - \text{var}(m_a)}\text{TE}$$

Second, managers face collateral requirements, which depend on the riskiness of the total position[3], and effectively limit their leverage. Assume, as an approximation, that a zero-price payoff requires a margin proportional to its standard deviation, and denote by c the margin required for a unit standard deviation. Thus, the manager will concentrate capital on the position which maximizes alpha per unit of standard deviation, and this position is precisely $m_b - m_a$. If the manager's capital is w, the alpha-maximizing position is:

$$x = \frac{w}{c}\frac{m_b - m_a}{\sqrt{\text{var}(m_b) - \text{var}(m_a)}}$$

and the corresponding maximum alpha is directly proportional to the available capital w, and inversely proportional to the margin requirement c:

$$\alpha_{\max} = \text{APR}_{\max}\frac{w}{c} = R_0\sqrt{\text{var}(m_b) - \text{var}(m_a)}\frac{w}{c}$$

In summary, the maximal alpha that can be achieved in practice is limited by institutional constraints, which imply directly or indirectly leverage bounds. With such bounds, the optimal strategy is to invest the maximum allowed amount in the payoff maximizing the appraisal ratio.

2.3 Exposure-switching strategies

The standard approach to performance evaluation entails the estimation of the regressions of the managed portfolio's excess returns on the benchmark excess returns. Can time- and performance-dependent variation in the exposure of the managed portfolio to the benchmark enhance estimated performance? Ideally, if benchmark excess returns are not predictable, switching exposure between periods should not generate alpha.

An exposure-switching managed portfolio policy may lead to a biased estimation of the portfolio's risk (beta) which in turn renders the regression misspecified. A misspecification of the regression can lead to a positive estimate of the regressions intercept which even in the absence of the ability to trade in anticipation of future returns would lead to the false inference that the portfolio manager has superior performance.

[3] As an example, since April 2007 the CBOE calculates the daily collateral requirement of a position in options with the same underlying as the maximum loss on the total position, assuming changes in the underlying asset ranging from -8% to +6% for a broad-based index and from -10% to 10% for a non broad-based index. See `http://www.sec.gov/rules/sro/cboe/2006/34-54919.pdf`

In a simulation, Goetzmann at al. (2007) report that switching exposure to a single-index benchmark can generate the appearance of an alpha although the benchmark returns are identically and independently distributed (IID). Under the assumption of IID returns, this section establishes a limit on the ability to bolster the fund performance by switching benchmark exposure between observation periods, and explains the alphas reported in the simulation by Goetzmann at al.

Two cases are considered in this section. In the first case the observation interval is fixed (e.g., a month), and the number of observations is very large, implying a very long time horizon T. It will be shown that switching risk exposure results in a *lower* appraisal ratio than the maximal one covered in Theorem 1. In the second case the time horizon T is fixed but the observation interval Δt is very small. It will be shown that alpha generated by switching risk exposure *deteriorates* as the time horizon T gets longer.

Consider a fund manager seeking to enhance the performance of a trading strategy by cleverly changing benchmark exposures over time. Let r_{mi} be the excess benchmark return during the time period $[(i-1)\Delta t, i\Delta t]$, with mean η and variance δ^2. The sequence $\{r_{mi}\}_{i=1,\cdots,n}$ is assumed IID. Let r_{zi} be the excess return of any payoff $z \in X_a$ for the time period $[(i-1)\Delta t, i\Delta t]$. Assume that the sequence $\{r_{zi}\}_{i=1,\cdots,n}$ is IID, and, without loss of generality, that the excess return r_{zi} is uncorrelated with the benchmark returns. Denote the mean of r_{zi} by a and its variance by h^2. The appraisal ratio of this strategy is a/h, and, by Theorem 1 is no greater than $\mathrm{APR}_{\max}$.

This subsection studies an attempt to further enhance the appraisal ratio of the strategy z by combining it with a strategy with time- and performance-dependent exposure to the benchmark. Thus, the total excess return of the fund takes the form of $r_{xi} = \beta_i r_{mi} + r_{zi}$, with β_i as a function of the past realizations of $\{\beta_j\}_{j=1,\cdots,i-1}$, $\{r_{mj}\}_{j=1,\ldots,i-1}$, and $\{r_{zj}\}_{j=1,\ldots,i-1}$.

With n periods of observed returns, the OLS regression of r_{xi} on r_{mi} produces an estimated alpha (denoted by $\hat{\alpha}_n$), an estimated beta (denoted by $\hat{\beta}_n$), and the appraisal ratio (denoted by $\widehat{\mathrm{APR}}_n$), which is the ratio of $\hat{\alpha}_n$ to the estimated tracking error $\hat{\sigma}_n$. The next theorem proves that in the long run switching benchmark exposures only increases tracking error, thereby worsening, not enhancing the appraisal ratio or the Sharpe ratio.

Theorem 2 *Assume that r_{mi} has a finite fourth moment and that $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \beta_i^k$ exists for $k \le 4$ and positive for $k = 2, 4$. Then, as n approaches to infinity, the alpha and the appraisal ratio converge to*

$$\lim_{n\to\infty} \hat{\alpha}_n = a \tag{16}$$

$$\lim_{n\to\infty} \widehat{\mathrm{APR}}_n = \frac{a}{\sqrt{\left(1+\frac{\eta^2}{\delta^2}\right)\left(h^2+\rho(\eta^2+\delta^2)\right)}} \tag{17}$$

where ρ is given as

$$\rho = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}(\beta_i - \bar{\beta}_n)^2 \quad \textit{with} \quad \bar{\beta}_n = \frac{1}{n}\sum_{i=1}^{n}\beta_i\,. \tag{18}$$

The theorem implies that $\lim \widehat{\mathrm{APR}}_n < a/h$, where a/h is in fact the appraisal ratio of the payoff z. This inequality shows that it is impossible to improve long-run performance by switching the benchmark exposure between observation periods if each period has fixed length Δt (e.g., a month or a quarter).

At variance with the preceding result, benchmark exposure switching can lead to the appearance of superior performance if the time horizon T is fixed and the observation interval Δt is very short, as it will be demonstrated in the next theorem. Suppose the price of the benchmark asset follows a geometric Brownian motion:

$$S_t = S_0 e^{(\mu - 0.5\sigma^2)t + \sigma B_t}\ , \tag{19}$$

where B_t is a Brownian motion. It follows from Ito's lemma that the asset's instantaneous return, $dX_t = dS_t/S_t$, follows the diffusion process

$$dX_t = \mu dt + \sigma dB_t\ . \tag{20}$$

If β_t is the benchmark exposure of the fund at time t, the instantaneous fund return satisfies

$$dY_t = \beta_t dX_t \tag{21}$$

Let n be the number of times to observe the excess returns of the benchmark and the fund and $\Delta t = T/n$ be the length of time between two observations. Regression of the observed fund return on the observed benchmark returns gives the intercept $\hat{\alpha}_n$ and slope $\hat{\beta}_n$. The next theorem shows how to choose β_t to maximize $\hat{\alpha}_n$ asymptotically.

Theorem 3 *Consider all the strategies with benchmark exposures β_t bounded between $\beta^{\min}$ and $\beta^{\max}$. Their maximal expected asymptotic alpha is*

$$\max E\left[\lim_{n\to\infty}\hat{\alpha}_n\right] = \sqrt{\frac{2}{9\pi T}}\ \sigma(\beta^{\max} - \beta^{\min})\ . \tag{22}$$

The maximum is achieved by a strategy that switches its benchmark exposure β_t^* between the lower and upper bounds:

$$\beta_t^* = \begin{cases} \beta^{\min}, & \textit{if } B_t \ge 0 \\ \beta^{\max}, & \textit{if } B_t < 0\,. \end{cases} \tag{23}$$

The strategy with the exposure β_t^* in equation (23) is a simple bang-bang strategy: choose the lowest feasible exposure to the benchmark when the cumulative excess return to date is positive, and the highest feasible exposure to the benchmark when the cumulative return is negative. The maximum expected asymptotic alpha in Theorem 3 is proportional to the volatility of the benchmark and the range of exposures. This is intuitive; higher volatility and range of exposures allow a larger payoff space to be generated from dynamic strategies.

The apparent similarity of Theorem 3 and Theorem 1 notwithstanding, they are different in several ways. Indeed, if the optimal strategy in Theorem 1 is achieved by delta trading, the strategy has dynamic exposure to the benchmark, similar to the strategy in Theorem 3. However, trading occurs at the same frequency of the observations in Theorem 3, whereas in Theorem 1 trading can be more frequent than observations. In addition, the observation interval Δt in Theorem 1 is fixed, whereas it shrinks to zero in Theorem 3. Moreover, the maximum expected asymptotic alpha in Theorem 3 is inversely proportional to the square-root of the time horizon T. As the time horizon increases, the alpha generated by switching benchmark exposure is pushed toward zero.

A strategy similar to Theorem 3 is devised by Goetzmann et al (2007). Choosing $\mu = 10\%$, $\sigma = 20\%$, $\beta^{\min} = 0.5$, $\beta^{\max} = 1.5$, and $T = 5$ years, they report an average alpha of 2.05% in 10,000 simulation runs. The average Sharpe ratio of their dynamic strategy is 0.673, slightly larger than the Sharpe ratio of 0.6 delivered by a simple investment strategy in the benchmark. Applying the same set of parameters to the right hand side of equation (22) gives

$$\max E[\lim_{n\to\infty} \hat{\alpha}_n] = \frac{5.32\%}{\sqrt{T}} = \frac{5.32\%}{\sqrt{5}} = 2.38\%\,, \tag{24}$$

which is not much higher than the average alpha obtained by Goetzmann at al., suggesting that their alpha is close to the maximum that can be generated by switching beta. Indeed, although the strategy in (23) is only optimal with continuous trading, its performance with discrete trading can be easily evaluated through simulation. Using the same monthly increments as in Goetzmann et al. (2007), from 100,000 simulation runs we obtain an average alpha of 2.33 percent (and a 95 percent confidence interval of (2.31, 2.35)), which confirms that continuous time asymptotics offer an accurate approximation for monthly observations, at least for typical parameter configurations. Thus, Theorem 3 indicates that the alpha in equation (24) cannot be expected for time horizons much longer than five years. For example, when $T = 10$, the alpha should be expected to be below $5.32\%/\sqrt{10} = 1.68\%$ on average.

This subsection demonstrates that exposure-switching strategies can create non-trivial alpha and that the strategy proposed by Goetzmann et al (2007) comes close to achieving the maximum alpha of the exposure-switching strategy. The following sections derive explicit formulas for the alpha- and appraisal ratio-maximizing policies.

3 Maximal Performance in a Complete Market

This Section studies the implications of Theorem 1 under the standard assumption that the benchmark assets follow a geometric Brownian motion. The first subsection offers explicit formulas for the maximal Sharpe and appraisal ratios in general, and then approximates these results for the limiting case that the observation interval is small. The intuition of this limiting case is transparent. The second subsection offers numerical examples, demonstrating that under the Black Scholes assumptions it would typically take a long time for a manager who has no superior information to produce a significantly positive alpha by augmenting his portfolio with options.

3.1 Theoretical results

As shown in equation (15), the maximal appraisal ratio can be calculated from the Sharpe ratios of X_a and X_b. The calculation of the Sharpe ratio of X_b is straightforward from the moments of the benchmark returns because

$$\text{SHP}_b = \sqrt{E[r'_m][\text{var}(r_m)]^{-1}E[r_m]} \ . \tag{25}$$

Similarly, if the attainable space X_a is spanned by a *finite* number of observable asset returns, it is straight forward to calculate the Sharpe ratio SHP_a from the moments of asset returns in X_a. The maximal appraisal ratio $\text{APR}_{\max}$ then follows from equation (15) as the square-root of the difference between SHP_a^2 and SHP_b^2.

As the space X_a may contain *infinitely* many security payoffs, the Sharpe ratio of X_a is more difficult to obtain than that of X_b. In this section, for a given benchmark space X_b, the attainable space X_a is assumed to be the set of all L^2-integrable functions of the benchmark payoffs. That is, $X_a = L^2(P, \Omega)$, where $L^2(P, \Omega)$ is the set of all squared-integrable functions of the benchmarks. In this case, the market of X_a is said to be complete, and m_a is the minimum-norm stochastic discount factor in $L^2(P, \Omega)$.

An important practical issue is the analysis of the payoff space which can be generated from trading derivatives on the benchmark assets. It is taken up next under the assumption that the prices of the benchmark assets follow a multivariate geometric Brownian motion. If the vector of the instantaneous annualized expected returns on the benchmark assets is $\mu = (\mu_1, \cdots, \mu_k)'$, under the assumption of geometric Brownian motion, the price of benchmark asset j at time t is

$$S_{j,t} = S_{j,0}e^{\left(\mu_j - 0.5\sigma_j^2\right)t + \sigma B_{jt}} \qquad 1 \le j \le k \ . \tag{26}$$

where B_{jt} is a Brownian motion, σ_j is its annualized volatility, and S_{j0} is the price at $t = 0$. The Brownian motions can be correlated. Let the correlation between B_{jt} and B_{lt} be ρ_{ij}. Then, $B_t = (B_{1t}, \cdots, B_{kt})' \sim N(0_k, t\Sigma)$, where the covariance matrix Σ has $\rho_{jl}\sigma_j\sigma_l$ as the element on the j^{th} row and l^{th} column and is assumed positive-definite. During the time interval $[t\,,\ t + \Delta t]$, where time is measured in years, the return on benchmark asset j is $R_j = S_{j,t+\Delta t}/S_{j,t}$. If the instantaneous annualized safe rate is denoted by r, the return on the safe asset during the same time interval is

$$R_0 = e^{r\Delta t}\,. \tag{27}$$

The parameters of the geometric Brownian motion are restricted by the moments of the benchmark excess returns. Since the j^{th} benchmark excess return is $r_j = R_j - R_0$, its expectation satisfies

$$E[r_j] = e^{r\Delta t}\left(e^{(\mu_j - r)\Delta t} - 1\right)\ , \quad j = 1, \cdots, k\ . \tag{28}$$

The variance of the j^{th} benchmark excess return satisfies

$$\text{var}(r_j) = e^{2r\Delta t}e^{2(\mu_j - r)\Delta t}\left(e^{\sigma_j^2\Delta t} - 1\right)\ , \quad j = 1, \cdots, k\ . \tag{29}$$

The covariance between r_j and r_l satisfies

$$\text{cov}(r_j, r_l) = e^{2r\Delta t}e^{[(\mu_j - r)+(\mu_l - r)]\Delta t}\left(e^{\rho_{jl}\sigma_j\sigma_l\Delta t} - 1\right) \qquad \text{for} \quad j, l = 1, \cdots, k; \quad j \neq l\ . \tag{30}$$

If the moments of the excess returns are given, equations (27)–(30) can be used to obtain the parameters in the geometric Brownian motion. The converse also holds: if the parameters of the geometric Brownian motion are given, these equations can be used to calculate the first two moments of r_m and then be substituted into equation (25) to obtain the Sharpe ratio of X_b.

Under the assumptions of geometric Brownian motion, which implies complete market, the maximal appraisal ratio can be obtained from equation (15) as derived in the Appendix. The result is as follows.

Theorem 4 *If the prices of the benchmark assets follow the geometric Brownian motion (26), the Sharpe ratio generated from derivatives or delta trading strategies is*

$$\text{SHP}_a = \sqrt{e^{(\mu - r1_k)'\Sigma^{-1}(\mu - r1_k)\Delta t} - 1}\,. \tag{31}$$

Once the Sharpe ratios of X_b and X_a are calculated, the maximal appraisal ratio can be obtained from equation (15), which becomes

$$\text{APR}_{\max} = \sqrt{e^{(\mu - r1_k)'\Sigma^{-1}(\mu - r1_k)\Delta t} - E[r_m'][\text{var}(r_m)]^{-1}E[r_m] - 1}\ , \tag{32}$$

where $E[r_m]$ and $\text{var}(r_m)$ can be calculated from equations (28)–(30). An important feature of the $\text{APR}_{\max}$ in equation (32) is that it approaches to zero if the observation interval Δt shrinks to zero or if the vector of risk premiums $\mu - r1_k$ drops to zero.

To see the exact relationship between the maximal appraisal ratio and the risk premium, the expression (32) is simplified. Without loss of generality, the benchmark excess returns can be chosen orthogonal to each other. In this case, $\rho_{jl} = 0$ for all $j \neq l$. Denote the annualized Sharpe ratio of the j^{th} benchmark by s_j, which equals $(\mu_j - r)/\sigma_j$, and let $s = (s_1, \cdots, s_k)'$. The maximal appraisal ratio becomes

$$\text{APR}_{\max} = \sqrt{e^{(s's)\Delta t} - \sum_{j=1}^{k} e^{-2s_j\sigma_j\Delta t}(e^{s_j\sigma_j\Delta t} - 1)^2(e^{\sigma_j^2\Delta t} - 1)^{-1} - 1} \ . \tag{33}$$

The first-order approximation of (33) in terms of Δt is

$$\text{APR}_{\max} = \left\{ s'[\Sigma + 2\text{Diag}(\mu - r1_k)]s + (s's)^2 \right\}^{1/2} \frac{\Delta t}{\sqrt{2}} \ + \ O(\Delta t^2) \ , \tag{34}$$

where $\text{Diag}(\mu - r1_k)$ is the $k \times k$ diagonal matrix in which the j^{th} diagonal element is $\mu_j - r$. In the case of a single benchmark (setting $k = 1$ and dropping off the subscripts of μ_1, σ_1 and s_1), the formula of the maximal appraisal ratio simplifies to

$$\text{APR}_{\max} = \sqrt{e^{s^2\Delta t} - e^{-2s\sigma\Delta t}(e^{s\sigma\Delta t} - 1)^2(e^{\sigma^2\Delta t} - 1)^{-1} - 1} \ , \tag{35}$$

and its first-order approximation is

$$\text{APR}_{\max} = (\mu - r + s^2)\frac{\Delta t}{\sqrt{2}} \ + \ O(\Delta t^2) \ . \tag{36}$$

According to the approximation, the maximal appraisal ratio is positively associated with the annualized risk premium $\mu_j - r$ and the annualized Sharpe ratio s_j of each benchmark.[4] It is also positively associated with time interval between the evaluator's two consecutive observations. This is reasonable because a longer observation period increases the manager's attainable space by delta trading or, equivalently, allows to more effectively exploit the non-linearity associated with the option payoffs.

The previous formulas remain valid in the presence of additional securities other than the benchmarks, which carry additional unpriced idiosyncratic risk. Put differently, using options on single securities cannot improve the significance of alpha, if the risk factors are already carried by the benchmarks, and the options on the benchmarks are used optimally.

[4] Figure 1 helps understand the relation between the appraisal ratio and the risk premium. Holding all other parameters fixed, an increase in the risk premium implies an increase in the slope of the dashed line representing the index return as well as an increase in the Sharpe ratio. Therefore the intercept of the dotted regression line will increase, thus showing a higher alpha. Since other parameters are being held constant, it follows that a higher alpha is associated with a higher appraisal ratio.

To see this fact, consider a market where, in addition to the k lognormal benchmarks $S_{j,t}$ for $1 \leq j \leq k$, there are n securities with excess returns r_i for $k+1 \leq i \leq n+k$, given by:

$$r_i = \sum_{j=1}^{k} \beta_{ij} r_j + \varepsilon_i$$

where ε_i for $k+1 \leq i \leq n+k$ are independent of the benchmarks and of each other, and $E[\varepsilon_i] = 0$. In other words, these securities carry unpriced specific idiosyncratic risks.

The discount factors m_a and m_b are the same as in the market with the benchmarks alone. To see this, it suffices to note that both discount factors price all the additional securities. This true for m_b because $E[\varepsilon_i] = 0$, and for m_a because ε_i is also uncorrelated with the benchmarks.

3.2 Numerical results

In performance evaluation the benchmark payoffs are often the factors used in the studies of the linear asset pricing models. The first factor is usually the market factor (MKT), which is the excess return on the value-weighted aggregate equity index compiled by the Center of Research on Security Prices (CRSP). Three additional factors are used by researchers, e.g., Fama and French (1993, 1996), to explain average stock returns: the firm-size factor (SMB) captures the firm-size premium documented by Banz (1981), the book-to-market factor (HML) captures the value premium presented by Stattman (1980), and the momentum factor (MOM) captures the profit of momentum trading strategy reported by Jegadeesh and Titman (1993). The monthly observations of these factors are available at Kenneth French's web site which also provides the returns on one-month Treasury bills (TBL). The sample of monthly excess returns on the factors from January of 1963 to December of 2006 are displayed in panel A of Table 1. Panel B reports the geometric Brownian motion parameters that are consistent with the sample moments. These parameters are solved from equations (27)–(30).

Consider three benchmark spaces: (1) X_b^{CA} is the benchmark space spanned by the MKT factor (CA alludes to the CAPM) and the safe return; (2) X_b^{FF} is the benchmark space spanned by the MKT, SMB and HML factors and the safe return; (3) X_b^{MM} is the benchmark space spanned by the MKT, SMB, HML and MOM factors and the safe return. The Sharpe ratios of these benchmark spaces can be calculated using equation (25) based on the parameters in Table 1 and various assumptions on the time interval Δt of observations. Column B of Table 2 reports the Sharpe ratios of the benchmark spaces for monthly ($\Delta t = 1/12$), quarterly ($\Delta t = 1/4$) and semiannual ($\Delta t = 1/2$) observations. As expected, a larger benchmark space or a longer observation period are associated with higher

Sharpe ratios. For example, the Sharpe ratio of X_b^{MM} is 0.631 for quarterly observation, higher than the ratio for monthly observation, which is 0.367. The quarterly Sharpe ratio of X_b^{MM} is also higher than the Sharpe ratio of X_b^{FF} for the same quarterly observation frequency, which is 0.463.

Let X_a^i be the payoff space that contains all the functions (with finite second moment) of the benchmarks in X_b^i, for $i =$ CA, FF, and MM. In the absence of derivative securities on these factors, the payoffs in X_a^{CA}, X_a^{FF} X_a^{MM} must be achieved by delta trading. The Sharpe ratios of the payoff spaces can be calculated using equation (31) and the parameters in Table 1. Column C of Table 2 reports the Sharpe ratios of the attainable spaces for monthly, quarterly and semi-annual observations. Again as expected, a larger attainable space or a longer observation period are associated with higher Sharpe ratios, and the Sharpe ratio of an attainable space is always higher than its corresponding benchmark space. For quarterly observations, as an example, the Sharpe ratio of X_a^{MM} is 0.708, higher than the Sharpe ratio of X_b^{FF}, which is 0.631.

By equation (15), the squared difference between the Sharpe ratios of the benchmark and attainable spaces gives the maximal appraisal ratio (namely, $\mathrm{APR}_{\max}$), which is reported in the column D of Table 2. In each observation frequency, the maximal appraisal ratio increases at least four fold when the SMB and HML factors are added to the benchmark space. When the MOM factor is further added to the benchmark space, the maximal appraisal ratio is nearly doubled. For example, in quarterly observations, $\mathrm{APR}_{\max}$ is 0.037 when the benchmark space is X_b^{CA} but increases to 0.171 when the benchmark space is enlarged to X_b^{FF}. It goes up to 0.322 when the MOM factor is added.

The t statistic of alpha can be approximated by the product of $\mathrm{APR}_{\max}$ and the square-root of the number of observations n. A significant t statistic at 95% confidence level requires $\sqrt{n} \times \mathrm{APR}_{\max} > 1.96$. Then, $n = (1.96/\mathrm{APR}_{\max})^2$ is approximately the minimum number of observations to obtain a significant t statistic. These numbers are reported in column E of Table 2.

Two patterns emerge from Table 2: as the benchmark space increases and as the observation period increases the appraisal ratio increases (and therefore the minimal number of years to obtain a significant t statistic decreases). The appraisal ratio increases because the risk premium and Sharpe ratio of the benchmark space increase as more indices are added to the benchmark space. Table 1 shows that HML and MOM factors have relatively high growth rates. Table 2 shows that inclusion of HML and MOM evidently increases the Sharpe ratio. As indicated by Theorem 4 and equation (36), the appraisal ratio is positively related to the risk premia of benchmark assets and the Sharpe ratio of the benchmark space.

Wang and Zhang (2003) anticipate these observations. Studying the frequency and magnitude with which pricing kernels associated with certain benchmarks take on negative values, they note that such negative values are relatively rare and small when the benchmark space is spanned by the market portfolio but become increasingly more frequent and larger as SMB and HML are added as benchmark portfolios. Negativity of the pricing kernel suggests that by trading options one can outperform the returns spanned in the benchmark space.

4 Performance Enhancement through Option Writing

Figure 1 suggests that the most intuitive way to appear to be out-performing is by writing options. This section explores this intuition, arguing that one can deliver the optimal policy exclusively by writing options. The next subsection applies Theorem 1 to describe the optimal option writing strategy. The relation between the pricing of the options and the process generating the benchmark returns is important for the assessment of the appraisal ratio that the optimal policy is likely to generate.

Subsection 4.1 studies these maximal appraisal ratios under the assumption that the S&P 500, Nasdaq 100 and Russell 2000 indices follow a geometric Brownian motion and that therefore the Black-Scholes formula correctly prices options on these indices with the volatility of the underlying geometric Brownian motion process. Under these assumptions it turns out that a fund manager who cannot predict future returns is unlikely to generate a statistically reliable alpha by trading derivatives.

Departures from geometric Brownian motion price evolution and the associated Black-Scholes option pricing have been studied, e.g., in Jones (2006) and Broadie, Chernov and Johannes (2007). Subsection 4.2 relaxes the geometric Brownian motion assumption and allows options to be priced at the actual implied volatility rather than at the volatility of the stochastic process governing the underlying asset. In this set up option payoffs cannot necessarily be replicated by delta trading and therefore options are not redundant assets. In this more general case it is possible to generate a statistically reliable alpha. A comparison between this more general case and the preceding one indicates that an alpha is likely to emerge even for the manager who cannot predict future returns not because of the non-linear nature of the payoff on the options but because of market incompleteness, i.e., because the option prices reflect departures from the assumptions in the Black-Scholes model.

4.1 When physical and implied volatilities coincide

Under the assumption of geometric Brownian motion, the solutions to the performance maximization problem (6) are combinations of the benchmark payoffs and the variable

$m_b - m_a$, which is a nonlinear function of the benchmark payoff. When the benchmark space consists of only one risky asset (in addition to the safe asset), the expression of the optimal payoff as function of the benchmark return R_m can be derived.

Theorem 5 *Assume that the benchmark space consists of only one risky asset and its price follows a geometric Brownian motion with growth rate μ and volatility σ. Assume the continuously-compounded safe rate is r. Then for any numbers γ and ϕ and any positive number θ the payoff satisfying*

$$x = \gamma + \phi R_m - \theta f(R_m) , \tag{37}$$

where

$$f(R_m) = cR_m^{-b} \quad \text{with} \quad b = (\mu - r)/\sigma^2 \; ; \; c = e^{[-r+0.5b(\mu+r-\sigma^2)]\Delta t} , \tag{38}$$

solves the optimization problem (6).

The payoff of the optimal strategy in equation (37) is a non-linear function of R_m because the function f is non-linear. Notice that the first derivative of the function is negative ($f' < 0$) and its second derivative is positive ($f'' > 0$). To show the nonlinear nature of the optimal strategies, panel A of Figure 2 displays the excess return on one-dollar investment in the strategy as a function of the rate of return on the benchmark index. The parameters in the stochastic process of the index are $\mu = 11.43\%$, $\sigma = 14.98\%$ and $r = 5.59\%$ per annum, which are the estimates for the market factor and T-bills rate in Table 1. The parameter θ is set to 1. The parameter ϕ is chosen so that the strategy's delta with respect to the benchmark is 1. To make x a return on one-dollar investment, γ is chosen so that the value of the strategy is 1. All the returns in the figure are annualized by setting $\Delta t = 1$.

Panel B of Figure 2 displays the excess return on one-dollar investment in the optimal strategy that is benchmark-neutral. In the benchmark-neutral strategy, ϕ is chosen so that its delta with respect to the benchmark is zero, implying $\text{cov}(r_x, r_m) \approx 0$. The parameter θ is again set to 1, and γ is chosen so that the value of the hedged strategy is 1. Notice that the excess return on the hedged strategy stays positive as long as the rate of return on the benchmark is between -13% and 17%. This means that the hedged strategy is profitable as long as the benchmark stays within the range around the current level. In practice, the profitable range should be narrower due to transaction costs and borrowing rates that are higher than the safe return. Losses will occur when the benchmark rises or drops outside of the range.

The optimal strategy can be implemented by writing options on the benchmarks because the nonlinear part $f(R_m)$ can be replicated by a portfolio of call and put options. In fact, integration by parts shows that for any $K > 0$ and any twice differentiable function f,

$$\begin{aligned} f(R_m) &= f(K) + f'(K)(R_m - K) \\ &\quad + \int_0^K f''(k)(k - R_m)^+ dk + \int_K^\infty f''(k)(R_m - k)^+ dk \ . \end{aligned} \tag{39}$$

The first and second integrations in equation (39) sum the payoffs of long positions in put and call options, respectively. Equations (37) and (39) indicate that the strategy that achieves the highest appraisal ratio involves writing a set of options.

If $K = 1$ and $\theta = 1$, the payoff of the performance maximizing strategy is

$$\hat{x} = \hat{\gamma} + \hat{\phi} R_m - \int_0^1 f''(k)(k - R_m)^+ dk - \int_1^\infty f''(k)(R_m - k)^+ dk \tag{40}$$

where $\hat{\gamma} = \gamma + (b-1)c$ and $\hat{\phi} = \phi + bc$. The first integration of equation (40) is over payoffs of out-of-the-money put options, and the second integration is over payoffs of out-of-the-money call options. Therefore, the optimal strategy can even be implemented by writing out-of-the-money options on the benchmark.

The strategy in equation (40) can be approximated by taking positions in options with discrete grids of moneyness k. For example, let $\Delta k = 1\%$ and $k_i = i\Delta k$. In this strategy the position in the call or put options out of the money by $100|1 - k_i|$ percent is $-f''(k_i)\Delta k$ units. For one-dollar investment in the strategy and the same parameters used in Figure 2, panel A of Figure 3 displays $-f''(k_i)\Delta k$. Notice that it is an increasing function of k_i. Thus, the strategy of equation (40) shorts more puts than calls. Also, the strategy shorts more puts when they are more out of the money. In contrast, the strategy shorts fewer calls that are more out of the money.

In the strategy in equation (40) the value of the short position on an option decreases fast as the option moves further out of the money. Use $h(k)$ to denote the price of a call (put) options on R_m with strike price $k \geq 1$ $(k < 1)$ and maturity Δt. Then, $h(k_i)f''(k_i)\Delta k$ is the premium collected from writing the call or put option that is out of the money by $100|1 - k_i|$ percent. If the parameters are chosen so that the value of the payoff $\hat{x}$ is one dollar, then $-h(k_i)f''(k)\Delta k$ is the portfolio weight on the options with moneyness k_i. The portfolio weights are plotted in panel B of Figure 3 which shows that in terms of option premiums collected, the strategy writes more options near the money than the options far out of the money.

The short leg of the performance-ratio maximization strategy implies that its delta, the sensitivity of the strategy's value to the index price, decreases as the time passes or as the index price rises. The delta of the strategy decreases as time passes because the delta of each call or put option is an increasing function of the time to maturity. Since each call or put option has a positive gamma, which is the sensitivity of delta to the index price, the strategy has a negative gamma, implying that its delta is a decreasing function of the index price. Figure 4 illustrates the delta of the performance-ratio maximization strategy for different levels of the index price and different time to maturity.

The maximal performance ratios relative to the S&P 500 (SPX), Nasdaq 100 (NDX), and Russell 2000 (RUT) indices are examined next because these indices often serve as benchmarks for evaluation of actively managed funds and options on these indices are actively traded. The daily realized index returns from February 23, 1985 to April 30, 2007, are used to estimate the relevant parameters. The resultant Sharpe ratios of the benchmark spaces and of the attainable spaces are reported in Table 3. The maximal appraisal ratios of the payoff spaces that contain all the derivatives on the benchmarks as well as the approximate number of years required to attain significant expected t statistics are also provided.

The attainable payoff spaces underlying the reported calculations are larger than the actual available derivative payoff spaces in that the calculations assume the availability of all derivatives. Actual index options are not available at all strike prices. Thus, the estimated Sharpe ratios of the attainable spaces and the appraisal ratio are biased upwards.

The expected alphas of the Sharpe ratio-maximizing strategies analyzed here are positive. Table 3, however, suggests that it should be difficult to generate *significant* alpha from trading options on the S&P 500, Nasdaq 100 and Russell 2000 indices. The results for MKT in Table 2 and those for SPX in Table 3 are similar because the two returns of the two indices are highly correlated. More generally, the messages of Tables 2 and 3 are similar. It will take thousands of years to generate significant alpha if the actively managed funds are evaluated with monthly returns. For example, with the SPX as the benchmark, 1803 years is the expected minimum time required to obtain a significant alpha by writing options on the SPX. It they are evaluated with quarterly or semiannual returns, it takes hundreds of years to generate significant alpha by trading derivatives. Therefore, the attempt to generate statistically significant alpha by trading index options is unlikely to succeed, so long as the benchmark assets follow a multivariate geometric Brownian motion and options are priced accordingly.

4.2 When physical and implied volatilities differ

The results reported in subsection 4.1 appear to be inconsistent with the practical experience in generating alpha by writing index options. Lo (2001) reports a high Sharpe ratio from a strategy which entails holding the S&P 500 index and writing put options. In a similar vein, Hill et al. (2006) report that a large alpha can be obtained from the strategy of buying the S&P 500 index and writing index call options. The attractiveness of the strategy led the

Chicago Board of Options Exchange (CBOE) to create the BuyWrite monthly index (BXM) to track the return from the strategy of writing one-month at-the-money call options every month. Indeed, the regression of monthly excess returns on the BXM index with respect to the S&P 500 delivers significant t statistics for the period from January 1990 to December 2005 (see Table 4). The performance of the BXM index is especially good in the first sub-period.

The results reported in Table 4 are based on actual option prices as oppose to *theoretical* Black-Scholes prices. Empirical evidence against the Black-Scholes model highlights a number of stylized facts. First, the implied volatility index VIX is consistently higher than the realized volatility of the underlying. Second, the implied volatility is higher for options more out-of-the-money (the so-called volatility "smile") and for lower strikes (the volatility "skew"). Third, both implied and realized volatilities exhibit significant time-variation across different strikes and maturities. The deviations from the Black-Scholes model as well as the nonlinearity of the payoffs may account for the excess performance of the BXM index. The rest of this subsection will show that the first stylized fact (i.e., the implied volatility being higher than the realized volatility) alone can account for a large part of the excess performance of the BXM index.

To apply the framework developed in this paper to the performance of the BXM index, one needs to identify the pricing kernels m_b and m_a. For simplicity, assume here a single risky benchmark asset whose return has a log-normal distribution. Options on it are priced according to the Black-Scholes formula, but possibly with a different implied volatility. Only options with single one-period maturity are considered and therefore there is no room for a term structure of implied volatilities. Moreover, all options of the same maturity are assumed to have the same implied volatility, thus abstracting from the volatility smile and skew. With the options having implied volatility different from that of the underlying asset, it is in general not possible to replicate the option payoffs by delta trading. Put differently, with these assumptions it is also assumed that the the benchmark assets do not complete the market, and therefore options are not redundant assets.

In the market considered here investment is made at time t and the return is received at time $t + \Delta t$. The continuously-compounded safe rate is r. The return on the risky asset follows a log-normal distribution, i.e., the return is

$$R_m = e^{(\mu - 0.5\sigma^2)\Delta t + \sigma\sqrt{\Delta t}\psi} \ . \tag{41}$$

where ψ is a standard normal random variable under the physical probability measure P. The parameters μ and σ are the physical growth rate and the physical volatility of the risky asset, respectively. It is important to notice the assumption of (41) does not imply that the asset price follows a geometric Brownian motion.

Derivative securities are also available in this market. The volatility according to which they are priced is the implied volatility, denoted by $\hat{\sigma}$. It is different from the physical volatility σ. Let $\lambda = \hat{\sigma}/\sigma$. Assume that all derivatives on the asset price with time-to-maturity Δt are priced by the implied volatility $\hat{\sigma}$. Then, the risk-neutral probability measure Q satisfies

$$E_Q[R_m] = e^{r\Delta t} \,, \tag{42}$$

$$\mathrm{var}_Q(\log R_m) = \hat{\sigma}^2 \Delta t \,. \tag{43}$$

Let the benchmark payoff space X_b be spanned by the safe return and the return on the risky asset. To evaluate the performance generated from trading derivatives, let X_a be the payoff space spanned by X_b and its derivative securities, i.e., X_a is spanned by all the functions $f(R_m)$ with finite second moments. The assumed completeness of X_a implies that m_a is the unique SDF and thus has the smallest-norm.

Theorem 6 *Under the assumptions in equations (41)–(43), if* $0 < \lambda < \sqrt{2}$*, then the maximal appraisal ratio is*

$$\begin{aligned} \mathrm{APR}_{\max}(\lambda) \;\; = \;\; & \Big[\lambda^{-1}(2-\lambda^2)^{-1/2} e^{\delta^2(2-\lambda^2)^{-1}\Delta t} \\ & - e^{-2s\sigma\Delta t}(e^{s\sigma\Delta t}-1)^2(e^{\sigma^2\Delta t}-1)^{-1} - 1\Big]^{1/2} \,. \end{aligned} \tag{44}$$

where $s = (\mu - r)/\sigma$ *and* $\delta = s + \sigma(\lambda^2 - 1)/2$.

When the implied volatility equals the physical volatility (i.e., $\lambda = 1$), the above equation reduces to equation (35). As the time interval Δt approaches zero, the maximum appraisal ratio approaches

$$\lim_{\Delta t \to 0} \mathrm{APR}_{\max}(\lambda) = \sqrt{\frac{1}{\lambda\sqrt{2-\lambda^2}} - 1} \,. \tag{45}$$

The above limit is 0 if $\lambda = 1$ and is an increasing function of λ for $\lambda \in [1, \sqrt{2})$. Therefore, when the implied volatility is higher than the actual volatility, trading derivatives can lead to a high appraisal ratio even when the observation interval is very short. It is however difficult to accomplish this when the implied and physical volatilities are the same.

Table 5 shows that the relatively high implied volatility during 1990–2005 offered an opportunity for the derivative trading strategy to perform well. During this period, the historical volatility estimated from the daily S&P 500 index is 16%, whereas the average VIX is 19%, giving $\lambda = 1.21$. Using equation (44) of Theorem 6, the maximal appraisal ratio of monthly returns during the period is $\sqrt{192} \times \mathrm{APR}_{\max} = 5.77$. This is much higher than the realized performance ratio of 2.20 reported in Table 4 for the BXM index, but it points in the right direction. Various explanations for the discrepancy between the theoretical and actual performance ratios are readily available. First, the strategy in BXM—buying S&P 500 index and selling at-the-money call options—does not necessarily achieve the maximal performance, whereas the strategy underlying the numbers in Table 5 is optimal. Second, the theoretical analysis ignores temporal variations in the implied and physical volatilities. Third, the realized performance is based on a finite sample instead of an infinite sample as assumed for $\text{APR}_{\max}$. These discrepancies between the assumptions in Theorem 6 and the data could lead to the difference between the numbers reported in Tables 4 and 5.

One pattern is common to Tables 4 and 5, namely that performance of the strategy was stronger in the earliest sub-period and deteriorated subsequently. Table 5 also shows that the opportunity offered by the high implied volatility declined over time. The ratio of implied-to-physical volatility was high ($\lambda = 1.39$) in the first five years of the sample in the table, and fell in later sub-periods. Correspondingly, the maximal appraisal ratio drops from 14.01 in the first sub-period to 1.48 in the last sub-period. The decline of the implied volatility relative to the physical volatility might have contributed to the weaker performance of the BXM during the last two sub-periods. Possibly the decline in the difference between implied and actual volatility reflects the entrance of speculators who sought to profit from this difference.

5 Concluding Remarks

The evaluation of actively managed funds amounts to addressing the following question: are the returns on the funds unusually good in comparison with those available by a portfolio of a given benchmark assets? Asset pricing theory states that if the space of payoffs spanned by the benchmark assets satisfies certain regularity conditions, then the question is equivalent to the question, are the returns on funds unusually good by the standard of the stochastic discount factor (SDF) which prices all the assets in the benchmark space?

This paper's original motivation is the observation that some commonly used SDFs take on negative values in some circumstances. Thus, they may price correctly the benchmark assets, but will price incorrectly derivative securities on the benchmark assets—even assign them negative prices. (A simple example is a security which pays one dollar when the SDF is negative and zero otherwise. Priced by the SDF, its price must be negative, which cannot be because the security entails no liability to its holder.)

This criticism is well known for linear asset pricing models. Typically, a linear pricing model delivers period-by-period arbitrage-free pricing of existing assets (and portfolios of these assets), given the factor structure of their returns. Dybvig and Ingersoll (1982) note the possible negativity of the SDF of the CAPM and study some of its implications. Grinblatt and Titman (1989) point out that the nonlinear value function distorts Jensen's alpha in the CAPM. They argue that valuation models should have positive state price densities.

Trading existing assets and derivatives on them are closely related. Famously, Black and Scholes (1973) and Merton (1973) show that trading of existing securities can replicate the payoffs of options on these securities. Therefore, one should be careful in interpreting excess returns of actively managed funds estimated from linear models because such funds trade rather than hold on to the same portfolios. Examples of interpretations of asset management techniques as derivative securities include Merton (1981) who argues that market-timing strategies are akin to option trading, Fung and Hsieh (2001) who report that hedge funds using trend-following strategies behave like a look-back straddle, and Mitchell and Pulvino (2001) who report that merger arbitrage funds behave like an uncovered put.

Motivated by the challenge of evaluating rule-based trading strategies, Glosten and Jagannathan (1994) suggest replacing the linear factor models with the Black-Scholes model. Wang and Zhang (2003) study the problem extensively and develop an econometric methodology to identify the problem in factor-based asset pricing models. They show that a linear model with many factors is likely to have large pricing errors over actively managed funds, because empirically the model delivers an SDF that allows for arbitrage over derivative-like payoffs.

Lo (2001) devotes a section to dynamic risk analysis in which he offers a series of numerical examples of how a fund can write options or equivalently trade the benchmark asset and thereby appear to have superior performance. Goetzmann et al. (2007) also study the ability of money managers to manipulate performance measures, and conclude that "a manager that seeks to manipulate many of the more popular measures can indeed produce very impressive performance statistics."

This paper considers a general problem of performance evaluation, focusing primarily on three closely related quantities: the appraisal ratio, the improvement of the Sharpe ratio relative to the highest Sharpe ratio from a static strategy which invests only in the benchmark assets, and the reliability of the estimated alpha (its t statistic).

The paper's basic result establishes: (i) a formula for the maximal appraisal ratio, assuming a constant investment opportunity set and an identical investment strategy across the observation intervals, a strategy which may include derivatives on the benchmark assets; (ii) the strategy which produces the maximal appraisal ratio. This formula shows the relation between the maximal improvement in the Sharpe ratio and the SDFs of the benchmark space and the larger space from which the manager picks his strategy's payoffs. The set of payoffs which delivers that maximal improvement in the Sharpe ratio is given in terms of the two SDFs.

The paper studies the possible magnitude of the maximum appraisal ratio by assuming that the manager has no private information and can trade only in the benchmark assets (as often as he wishes) and derivatives on them. Applying the basic result to a set-up in which the benchmark asset prices follow a geometric Brownian motion with parameters matching those estimated from familiar index returns (e.g., the S&P 500 returns), a money manager who uses options optimally will only minimally enhance the measured Sharpe ratio of his fund if the options are priced according to the Black-Scholes model. Another aspect of the basic result is that the alpha generated by this manager will be statistically indistinguishable from zero unless he is followed by the evaluator for many years.

Options with prices which are at variance with the Black-Scholes model may open the possibility of enhancing the Sharpe and appraisal ratios. But this is due not merely to the non-linear nature of the options payoff, but to the violation of the assumption of geometric Brownian motion which is necessary for the validity of the Black-Scholes formula.

A Appendix

A.1 Proof of Theorem 1

As shown in Hansen and Jagannathan (1991) the stochastic discount factor m_b satisfies

$$m_b = \frac{1}{R_0} - \frac{1}{R_0} E[r_m'](\mathrm{var}(r_m))^{-1}(r_m - E[r_m]) \ . \tag{A1}$$

It follows from equations (2), (3) and (A1) that the asymptotic alpha can be written as $\alpha(x) = E[r_x m_b]R_0$, which gives equation (7) in Theorem 1 because $E[r_x m_a] = 0$.

Being projected on r_m and $m_b - m_a$, the return r_x can be decomposed as

$$r_x = \gamma + r_m' \beta + (m_b - m_a)\theta + \varepsilon \tag{A2}$$

where ε is the residual of the projection. It follows that $E[\varepsilon] = E[(m_b - m_a)\varepsilon] = 0$ and $E[r_m \varepsilon] = 0_k$. Since $m_b - m_a$ and r_m are uncorrelated, β satisfies equation (3). In view of equation (7), the decomposition gives

$$\begin{aligned} \alpha(x) &= R_0 E[(\gamma + r_m' \beta + \theta(m_b - m_a) + \varepsilon)(m_b - m_a)] \\ &= \theta \|m_b - m_a\|^2 R_0 \ . \end{aligned} \tag{A3}$$

Once $\|m_b - m_a\|$ is nonzero, $\alpha(x)$ is positive if and only if $\theta > 0$.

The decomposition in equation (A2) also gives

$$\mathrm{var}(r_x - r_m' \beta) = \theta^2 \|m_b - m_a\|^2 + \|\varepsilon\|^2 \ . \tag{A4}$$

It then follows from equations (4), (A3) and (A4) that the appraisal ratio is

$$\mathrm{APR}(x) = \frac{\theta \, \|m_b - m_a\|^2 R_0}{\sqrt{\theta^2 \|m_b - m_a\|^2 + \|\varepsilon\|^2}} \ . \tag{A5}$$

It has an upper bound $\|m_b - m_a\|$ because

$$\frac{|\theta| \cdot \|m_b - m_a\|^2 R_0}{\sqrt{\theta^2 \|m_b - m_a\|^2 + \|\varepsilon\|^2}} \le \|m_b - m_a\| R_0 \,. \tag{A6}$$

The upper bound is achieved when $\|\varepsilon\| = 0$. For any $z \in X_b$ and $\theta > 0$, the payoff $x = z + \theta(m_b - m_a)$ has zero residual in projection (A2) and thus its appraisal ratio $\mathrm{APR}(x)$ achieves the upper bound. This gives equation (9) in Theorem 1. It follows that $\max\{\mathrm{APR}(x) : x \in X_a\} = \|m_b - m_a\| R_0$, which is equation (8) in Theorem 1.

A.2 Proof of Theorem 2

The assumption in the theorem implies the existence of number b satisfying equation (18) and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \beta_i = b \tag{A7}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \beta_i^2 = b^2 + \rho \,. \tag{A8}$$

Let $\hat{\sigma}_n$ be the estimated tracking error, which is the product of $\sqrt{n}$ and the standard error of the estimated alpha. To prove the theorem, it is sufficient to derive

$$\lim_{n\to\infty} \hat{\alpha}_n = a \tag{A9}$$

$$\lim_{n\to\infty} \hat{\sigma}_n = \left(1 + \frac{\eta^2}{\delta^2}\right) (h^2 + \rho(\eta^2 + \delta^2)) \tag{A10}$$

because equation (17) will follow immediately from (A9) and (A10), since $\widehat{\mathrm{APR}}_n = \alpha_n / \hat{\sigma}_n$.

To show equation (A9), the following limits are shown first:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} r_{mi} = \eta \tag{A11}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} r_{mi}^2 = \eta^2 + \delta^2 \tag{A12}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} r_{xi} = a + b\eta \tag{A13}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} r_{mi} r_{xi} = a\eta + b(\eta^2 + \delta^2) \,. \tag{A14}$$

These equations give

$$\begin{aligned} \lim_{n\to\infty} \hat{\beta}_n &= \frac{\frac{1}{n}\sum_{i=1}^{n} r_{mi} r_{xi} - (\frac{1}{n}\sum_{i=1}^{n} r_{mi})(\frac{1}{n}\sum_{i=1}^{n} r_{xi})}{\frac{1}{n}\sum_{i=1}^{n} r_{mi}^2 - (\frac{1}{n}\sum_{i=1}^{n} r_{mi})^2} \\ &= \frac{b(\eta^2 + \delta^2) - \eta^2 b}{\eta^2 + \delta^2 - \eta^2} = b \,. \end{aligned} \tag{A15}$$

Equation (A9) can then be obtained from equations (A11), (A13) and (A15) as

$$\begin{aligned}\lim_{n\to\infty}\hat{\alpha}_n &= \lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n} r_{xi} - \hat{\beta}_n\frac{1}{n}\sum_{i=1}^{n} r_{mi}\right)\\ &= (a+b\eta) - b\eta = a\,. \end{aligned} \tag{A16}$$

Note that equations (A11) and (A12) immediately follow from the law of large numbers. The derivation of equation (A13) applies a Lemma from Section 12.14 in Williams (1991), which is a version of the strong law of large numbers for martingales:

> **Lemma**: *Let $(M_n)_{n\geq 0}$ be a square-integrable martingale, and denote by $A_n = \sum_{i=1}^{n} E_{i-1}[(M_i - M_{i-1})^2]$. If $A_n \uparrow \infty$ a.s., then $M_n/A_n \to 0$ a.s.*

To derive equation (A13), consider

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n} r_{xi} &= \frac{1}{n}\sum_{i=1}^{n}\beta_i r_{mi} + \frac{1}{n}\sum_{i=1}^{n} r_{zi}\\ &= \frac{1}{n}\sum_{i=1}^{n}\beta_i(r_{mi}-\eta) + \eta\frac{1}{n}\sum_{i=1}^{n}\beta_i + \frac{1}{n}\sum_{i=1}^{n} r_{zi}\,. \end{aligned} \tag{A17}$$

The second term converges to $b\eta$ by equation (A7), and the last one converges to a by the law of large numbers. For the first term, observe that $M_n = \sum_{i=1}^{n}\beta_i(r_{mi}-\eta)$ is an L^2 martingale, and that $A_n = \sum_{i=1}^{n} E_{i-1}[(M_i - M_{i-1})^2] = \delta^2\sum_{i=1}^{n}\beta_i^2$. Hence, the lemma implies $\lim_{n\to\infty} M_n/A_n = 0$. Since

$$\begin{aligned}\lim_{n\to\infty}\frac{M_n}{A_n} &= \lim_{n\to\infty}\frac{\sum_{i=1}^{n}\beta_i(r_{mi}-\eta)}{\delta^2\sum_{i=1}^{n}\beta_i^2}\\ &= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\beta_i(r_{mi}-\eta)\,\lim_{n\to\infty}\frac{1}{\delta^2\frac{1}{n}\sum_{i=1}^{n}\beta_i^2}\,, \end{aligned} \tag{A18}$$

it follows from equation (A8) that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\beta_i(r_{mi}-\eta) = 0\,. \tag{A19}$$

In a similar fashion, one can obtain equation (A14) by writing

$$\frac{1}{n}\sum_{i=1}^{n} r_{mi}r_{xi} = \frac{1}{n}\sum_{i=1}^{n}\beta_i(r_{mi}^2 - E[r_{mi}^2]) + (\eta^2+\delta^2)\frac{1}{n}\sum_{i=1}^{n}\beta_i + \frac{1}{n}\sum_{i=1}^{n} r_{mi}r_{zi}\,, \tag{A20}$$

which employed $E[r_{mi}^2] = \eta^2 + \delta^2$. The second term converges to $b(\eta^2+\delta^2)$, and the third one converges to $a\eta$ by the law of large numbers and the assumption that r_{mi} and r_{zi} are uncorrelated. The assumption of a finite fourth moment ensures that the martingale in the first term is square integrable. Thus, it converges to zero by the lemma.

Now turn to equation (A10), which requires calculating the limit of

$$\hat{\sigma}_n^2 = \frac{\sum_{i=1}^n (r_{xi} - \hat{\alpha}_n - \hat{\beta}_n r_{mi})^2}{n-2} \frac{\frac{1}{n}\sum_{i=1}^n r_{mi}^2}{\frac{1}{n}\sum_{i=1}^n r_{mi}^2 - (\frac{1}{n}\sum_{i=1}^n r_{mi})^2} . \tag{A21}$$

For the second factor, (A11) and (A12) imply that:

$$\lim_{n\to\infty} \frac{\frac{1}{n}\sum_{i=1}^n r_{mi}^2}{\frac{1}{n}\sum_{i=1}^n r_{mi}^2 - (\frac{1}{n}\sum_{i=1}^n r_{mi})^2} = \left(1 + \frac{\eta^2}{\delta^2}\right) . \tag{A22}$$

For the sum of squared regression residuals, write:

$$\begin{aligned} &\frac{1}{n}\sum_{i=1}^n (r_{xi} - \hat{\alpha}_n - \hat{\beta}_n r_{mi})^2 \\ &\quad = \frac{1}{n}\sum_{i=1}^n (\hat{\alpha}_n^2 - 2\hat{\alpha}_n r_{xi} + 2\hat{\alpha}_n\hat{\beta}_n r_{mi} + r_{xi}^2 + \hat{\beta}_n^2 r_{mi}^2 - 2\hat{\beta}_n r_{mi} r_{xi}) \\ &\quad = \hat{\alpha}_n^2 - 2\hat{\alpha}_n \frac{1}{n}\sum_{i=1}^n r_{xi} + 2\hat{\alpha}_n\hat{\beta}_n \frac{1}{n}\sum_{i=1}^n r_{mi} \\ &\qquad + \hat{\beta}_n^2 \frac{1}{n}\sum_{i=1}^n r_{mi}^2 - 2\hat{\beta}_n \frac{1}{n}\sum_{i=1}^n r_{mi} r_{xi} + \frac{1}{n}\sum_{i=1}^n r_{xi}^2 . \end{aligned} \tag{A23}$$

By the previously established limits, the first term converges to a^2, the second to $-2a(a+b\eta)$, the third to $2ab\eta$, the fourth to $b^2(\eta^2+\delta^2)$ and the fifth to $-2b(a\eta + b(\eta^2+\delta^2))$. For the sixth term, one has

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n r_{xi}^2 = a^2 + h^2 + 2ab\eta + (\eta^2+\delta^2)(\rho + b^2) . \tag{A24}$$

To see this, expand it as:

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^n r_{xi}^2 \quad &= \quad \frac{1}{n}\sum_{i=1}^n r_{zi}^2 + 2\frac{1}{n}\sum_{i=1}^n \beta_i E[r_{mi} r_{zi}] + \frac{1}{n}\sum_{i=1}^n \beta_i^2 E[r_{mi}^2] \\ &\quad + \frac{1}{n}\sum_{i=1}^n 2\beta_i (r_{mi} r_{zi} - E[r_{mi} r_{zi}]) + \frac{1}{n}\sum_{i=1}^n \beta_i^2 (r_{mi}^2 - E[r_{mi}^2]) . \end{aligned} \tag{A25}$$

The last two terms converge to zero by the lemma. The first term converges to $a^2 + h^2$ by the law of large numbers. The second term converges to $b\eta a$, and the third term converges to $(\eta^2+\delta^2)(b^2+\rho)$. Summing up, it follows that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n (r_{xi} - \alpha_n - \beta_n r_{mi})^2 = h^2 + \rho(\eta^2+\delta^2) , \tag{A26}$$

which gives equation (A10).

A.3 Proof of Theorem 3

The benchmark and portfolio returns during the time interval $[(i-1)\Delta t,\ i\Delta t]$ are

$$x_i = \mu\Delta t + \sigma(B_{i\Delta t} - B_{(i-1)\Delta t}) \tag{A27}$$

$$y_i = \int_{(i-1)\Delta t}^{i\Delta t} \beta_t dX_t \,. \tag{A28}$$

The estimated beta in the OLS regression of y_i on x_i is

$$\hat{\beta}_n = \frac{\sum_{i=1}^n x_i y_i - (\frac{1}{n}\sum_{i=1}^n x_i)(\sum_{i=1}^n y_i)}{\sum_{i=1}^n x_i^2 - (\frac{1}{n}\sum_{i=1}^n x_i)(\sum_{i=1}^n x_i)}\,, \tag{A29}$$

and the estimated alpha is

$$\hat{\alpha}_n = \frac{1}{n}\sum_{i=1}^n y_i - \hat{\beta}_n \frac{1}{n}\sum_{i=1}^n x_i\,. \tag{A30}$$

Let $\alpha_T = \lim_{n\to\infty} \hat{\alpha}_n$ and $\beta_T = \lim_{n\to\infty} \hat{\beta}_n$. The first task is to show

$$\hat{\beta}_T = \frac{1}{T}\int_0^T \beta_t dt \tag{A31}$$

$$\hat{\alpha}_T = \frac{\sigma}{T}\left(\int_0^T \beta_t dt - B_T \hat{\beta}_T\right)\,. \tag{A32}$$

It follows from equations (A27) and (A28) that $\sum_{i=1}^n x_i = \frac{1}{n}(\mu T + B_T)$, and $\frac{1}{n}\sum_{i=1}^n x_i$ converges to zero as n increases to infinity. By the same token, $\sum y_i$ converges to $\int_0^T \mu\beta_t dt + \int_0^T \beta_t \sigma dB_t$ and $\frac{1}{n}\sum_{i=1}^n y_i$ converges to zero. One also has

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n x_i^2 = \lim_{n\to\infty}\sum_{i=1}^n (\mu\Delta t + \sigma(B_{i\Delta t} - B_{(i-1)\Delta t}))^2 = \sigma^2 T \tag{A33}$$

and, similarly,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n x_i y_i = \sigma^2 \int_0^T \beta_t dt\,. \tag{A34}$$

Applying these results to equation (A29) proves equation (A31). When $n \to \infty$, the estimated alpha in equation (A29) gives

$$\hat{\alpha}_T = \frac{Y_T}{T} - \hat{\beta}_T \frac{X_T}{T} = \sigma\left(\frac{1}{T}\int_0^T \beta_t dB_t - \hat{\beta}_T \frac{B_T}{T}\right)\,, \tag{A35}$$

which implies equation (A32).

The expected alpha of a trading strategy is given by:

$$E[\hat{\alpha}_T] = -\frac{\sigma}{T^2} E\left[\int_0^T \beta_t B_T dt\right] = -\frac{\sigma}{T^2}\int_0^T E[\beta_t B_T]dt = -\frac{\sigma}{T^2}\int_0^T E[\beta_t B_t]dt \tag{A36}$$

Thus, to maximize this quantity while leaving $\beta_t \in [\beta_{min}, \beta_{max}]$ at all times, one needs to minimize $E[\beta_t B_t]$. This is the same as $\mathrm{cov}(\beta_t, B_t)$ since B_t has zero mean. This minimum is attained for

$$\hat{\beta}_s = \begin{cases} \beta^{min} & \text{if } B_s \geq 0 \\ \beta^{max} & \text{if } B_s < 0 \end{cases} , \tag{A37}$$

and therefore

$$E[\beta_t B_t] = \beta^{min} E[B_t | B_t > 0]/2 + \beta^{max} E[B_t | B_t < 0]/2 . \tag{A38}$$

Since

$$E[B_t | B_t > 0] = 2 \int_0^\infty u \frac{e^{-\frac{u^2}{2t}}}{\sqrt{2\pi t}} du = \sqrt{\frac{2t}{\pi}} , \tag{A39}$$

it follows that

$$E[\beta_t B_t] = -(\beta^{max} - \beta^{min}) \sqrt{\frac{t}{2\pi}} \tag{A40}$$

and hence

$$E[\hat{\alpha}_T] = \frac{\sigma}{T^2} \int_0^T (\beta^{max} - \beta^{min}) \sqrt{\frac{t}{2\pi}} dt = \frac{\sigma}{\sqrt{T}} (\beta^{max} - \beta^{min}) \frac{1}{3} \sqrt{\frac{2}{\pi}} , \tag{A41}$$

which proves equation (22).

A.4 Proof of Theorem 4

Under the assumption of geometric Brownian motion, the unique (and hence the smallest-norm) discount factor m_a is the discounted Radon-Nikodym density of the risk-neutral probability Q with respect to the physical probability:

$$m_a = e^{-[r + 0.5(\mu - r1_k)' \Sigma^{-1} (\mu - r1_k)] \Delta t - (\mu - r1_k)' \Sigma^{-1} \Delta B} , \tag{A42}$$

where ΔB is the change of the Brownian motion from t to $t + \Delta t$, and 1_k is a k-dimensional vector with all components equal to 1 (see Karatzas and Shreve, 1998, section 1.5). Equation (A42) is the key to the solution of the fund manager's maximization problem. With this equation, the variance of the stochastic discount factor is calculated as

$$\mathrm{var}(m_a) = e^{-2r\Delta t} \left(e^{(\mu - r1_k)' \Sigma^{-1} (\mu - r1_k) \Delta t} - 1 \right) . \tag{A43}$$

A substitution this equation into (14) implies that the Sharpe ratio of X_a is

$$R_0 \sqrt{\mathrm{var}(m_a)} , \tag{A44}$$

which gives equation (31).

A.5 Proof of Theorem 5

Under the assumption of geometric Brownian motion, the solutions to the performance maximization problem (6) are linear combinations of the benchmark payoffs and the variable $m_b - m_a$. The expression of the optimal solution can be derived in the case of a single benchmark return R_m. In this case, the stochastic discount factor is

$$\begin{aligned} m_a &= e^{-[r+0.5(\mu-r)^2/\sigma^2]\Delta t-[(\mu-r)/\sigma^2]\sigma\Delta B} \\ &= e^{-[(\mu-r)/\sigma^2]\{[\mu-0.5\sigma^2]\Delta t+\sigma\Delta B\}+\{-r\Delta t+0.5(\mu-r)(\mu+r-\sigma^2)/\sigma^2\}\Delta t} . \end{aligned} \quad \text{(A45)}$$

It follows from equations (26) and (A45) that $m_a = f(R_m)$ where $f(R_m)$ is defined in (38). The optimal strategy also involves m_b, which must be a linear function of R_m, i.e., $m_b = a + a_m R_m$ for some constants a and a_m. It follows from equation (9) in Theorem 1 that the payoff in equation (37) is an optimal strategy to the maximization problem (6).

A.6 Proof of Theorem 6

The smallest-norm discount factor m_b is given by (A1) and satisfies

$$\text{var}(m_b) = e^{-2(\mu+r)\Delta t}(e^{\mu\Delta t} - e^{r\Delta t})^2(e^{\sigma^2\Delta t} - 1)^{-1} . \quad \text{(A46)}$$

The unique stochastic discount factor (SDF) in X_a is the Radon-Nikodym density of the risk-neutral probability Q with respect to the physical probability P, divided by the safe return. The expression for the SDF is

$$\begin{aligned} m_a &= \lambda^{-1}e^{-r\Delta t+0.5\psi^2-0.5\lambda^{-2}(\psi+\delta\sqrt{\Delta t})^2} , \\ &\text{where} \quad \delta = (\mu - r)/\sigma + 0.5\sigma(\lambda^2 - 1) . \end{aligned} \quad \text{(A47)}$$

This is also the smallest-norm discount factor in X_a. The variance of m_a is finite and satisfies

$$\text{var}(m_a) = e^{-2r\Delta t}\left(\lambda^{-1}(2-\lambda^2)^{-1/2}e^{\delta^2\Delta t/(2-\lambda^2)} - 1\right) , \quad \text{(A48)}$$

provided that $\lambda \le \sqrt{2}$. It follows by equation (12) that the maximal appraisal ratio is (44).

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B Tables and Figures

B.1 Tables

Table 1: *Estimated Parameters for the Factors.* Panel A reports the sample means, standard deviations, covariances, correlations and Sharpe ratios of the monthly excess returns on the MKT, SMB, HML, and MOM factors from January of 1963 to December of 2006. It also reports the average monthly Treasury bill rates. Variable definitions are in the text. The data of the monthly excess returns are from the web site of Kenneth French. Panel B presents the geometric Brownian motion parameters consistent with the sample means and covariances of the monthly returns. These parameters are annualized and solved from equations (27)–(30).

A. Sample Moments of Monthly Returns

	TBL	MKT	SMB	HML	MOM
Mean	0.47%	0.49%	0.25%	0.47%	0.82%
Standard deviation		1.26%	0.93%	0.83%	1.15%
Sharpe ratio		0.112	0.077	0.162	0.205
Correlation	MKT		0.301	-0.409	-0.064
	SMB			-0.280	0.022
	HML				-0.115

B. Parameters in Geometric Brownian Motion

	TBL	MKT	SMB	HML	MOM
Mean	0.47%	0.49%	0.25%	0.47%	0.82%
Standard deviation		1.26%	0.93%	0.83%	1.15%
Sharpe ratio		0.112	0.077	0.162	0.205
Correlation	MKT		0.301	-0.409	-0.064
	SMB			-0.280	0.022
	HML				-0.115

Table 2: *Performance with Factors as Benchmarks.* Three sets of benchmark assets (and their corresponding payoff spaces) are considered: (i) the safe asset and MKT, (ii) the safe asset, MKT, SML and HML, and (iii) the safe asset, MKT, SML, HML and MOM. Variable definitions are in the text. Three observation frequencies are considered: monthly, quarterly and semi-annual. Using the data and parameters in Table 1, for each benchmark set and each observation period, the following are estimated: Sharpe ratio of the benchmark space, Sharpe ratio of the space generated by the benchmark space and delta trading on its members, the maximal appraisal ratio of the space according to Theorem 4, the approximate minimum number of years it would take to obtain a significantly positive alpha t-statistics in the regression of the returns of the optimal delta trading strategy on returns of the benchmark assets, and the annualized maximal alpha when constraining the annualized tracking error to be no more than 10 percent.

A	B	C	D	E	F
Factors spanning the benchmark space	Sharpe ratio of the benchmark	Sharpe ratio of the attainable	Maximal appraisal ratio	Minimal years for significance	Maximal alpha (annualized)
		Monthly Observations			
MKT	0.112	0.113	0.012	2084	0.43%
MKT, SMB, HML	0.269	0.275	0.056	103	1.93%
MKT, SMB, HML, MOM	0.367	0.381	0.103	30	3.56%
		Quarterly Observations			
MKT	0.193	0.197	0.037	694	0.74%
MKT, SMB, HML	0.463	0.494	0.171	33	3.42%
MKT, SMB, HML, MOM	0.631	0.708	0.322	9	6.44%
		Semi-annual Observations			
MKT	0.271	0.281	0.074	346	1.05%
MKT, SMB, HML	0.649	0.739	0.354	15	5.00%
MKT, SMB, HML, MOM	0.883	1.120	0.689	4	9.75%

Table 3: *Performance Ratios with Market Indices as Benchmarks.* Three sets of benchmark assets (and their corresponding payoff spaces) are considered: (i) the safe asset and SPX (the S&P 500) (ii) the safe asset, SPX, NDX (the Nasdaq 100), and (iii) the safe asset, SPX, NDX, and RUT (Russell 2000). Variable definitions are in the text. Three observation frequencies are considered: monthly, quarterly and semi-annual. For each benchmark set and each observation period, the following are estimated: Sharpe ratio of the benchmark space, Sharpe ratio of the space generated by the benchmark space and delta trading on its members, the maximal appraisal ratio of the space according to Theorem 4, the approximate minimum number of years it would take to obtain a significantly positive alpha t-statistics in the regression of the returns of the optimal delta trading strategy on returns of the benchmark assets, and the annualized maximal alpha when constraining the annualized tracking error to be no more than 10 percent. The data of the indices are from Bloomberg.

A	B	C	D	E	F
Indices spanning the benchmark space	Sharpe ratio of the benchmark	Sharpe ratio of the attainable	Maximal appraisal ratio	Minimal years for significance	Maximal alpha (annualized)
		Monthly Observations			
SPX	0.1174	0.1182	0.0133	1803	0.46%
SPX, NDX	0.1257	0.1268	0.0167	1148	0.58%
SPX, NDX, RUT	0.1276	0.1288	0.0174	1052	0.60%
		Quarterly Observations			
SPX	0.2022	0.2061	0.0400	600	0.80%
SPX, NDX	0.2157	0.2214	0.0500	384	1.00%
SPX, NDX, RUT	0.2187	0.2249	0.0522	352	1.04%
		Semi-annual Observations			
SPX	0.2835	0.2946	0.0802	299	1.13%
SPX, NDX	0.3009	0.3170	0.0997	193	1.41%
SPX, NDX, RUT	0.3047	0.3220	0.1041	177	1.47%

Table 4: *The Alpha of the BXM Index, 1990-2005.* The alpha and the corresponding t-statistic of the BXM index are estimated by OLS regression of the monthly excess returns of the BXM index on the monthly excess returns of the S&P 500 index. The data of the monthly indices are from Bloomberg.

Period	Annualized average return of S&P 500	Annualized average return of BXM	Annualize alpha of BXM over S&P 500	t Statistics of alpha of BXM over S&P 500
1990.01 - 2005.12	7.07%	6.82%	2.66%	2.20
1990.01 - 1994.12	4.50%	6.55%	4.11%	2.59
1995.01 - 1999.12	21.36%	14.32%	2.41%	0.90
2000.01 - 2005.12	-2.70%	0.80%	2.53%	1.22

Table 5: *Maximal Appraisal Ratios with Implied Volatility Possibly Greater than Realized Volatility.* The maximal appraisal ratios are calculated from equation (44). The historical volatility is estimated from the daily observations of the S&P 500 index for the each of the periods indicated in the table. The estimated implied volatility is the daily average of VIX for the corresponding period. The ratio of estimated implied volatility to the estimated historical volatility is the λ used in equation (44). The maximal alpha are computed under the constraint that the tracking error cannot exceed 10 percent. The data of the S&P 500 index and VIX are from Bloomberg.

Period	Historical volatility of S&P 500	Implied volatility (VIX)	Ratio of implied to historical	Maximal appraisal ratio	Maximal alpha
1990.01 - 2005.12	16%	19%	1.21	5.77	4.16%
1990.01 - 1994.12	12%	17%	1.39	14.01	18.08%
1995.01 - 1999.12	16%	20%	1.27	7.96	10.28%
2000.01 - 2005.12	19%	21%	1.11	1.48	1.75%

B.2 Figures

Figure 1: *Generating Positive Alpha by Writing Options.* A fund which is completely invested in the market index has a return equal to the market's return (the dashed line) and zero intercept. A fund which writes an index call option and invests the proceeds in the safe asset will have a return which is sensitive to the market's return in a nonlinear fashion (the dotted-dashed line). A fund which invests in the index and writes call options will also have a return which is sensitive to the market's return in a nonlinear fashion (the thick line). That return has a positive intercept in its regression on the market's return (the dotted line). Figure reprinted from Wang and Zhang (2003).

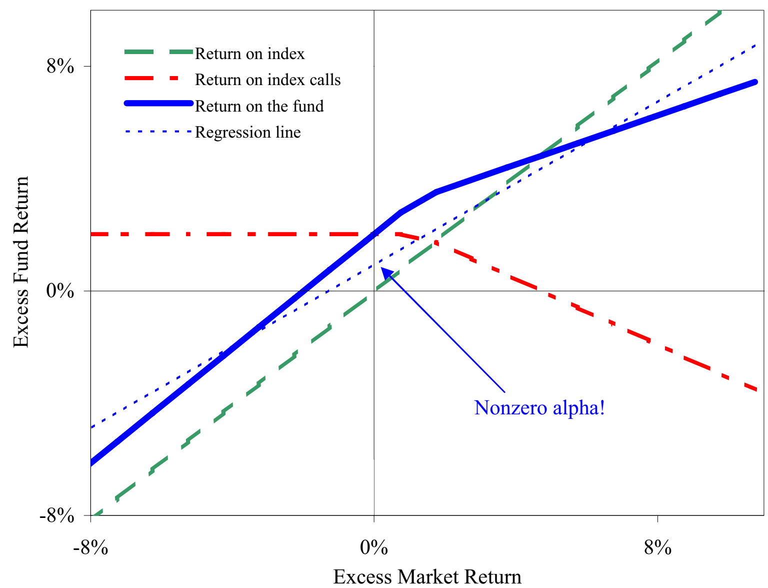

Figure 2: *The Nonlinear Returns on Optimal Strategy.* Panel A displays the excess return on one-dollar investment in the optimal strategy in equation (37) as a function of the rate of return on the benchmark. Panel B displays the excess return on one-dollar investment in the hedged optimal strategy. The parameters in the stochastic process of the benchmark are $\mu = 11.43\%$, $\sigma = 14.98\%$ and $r = 5.59\%$ per annum. The parameter θ is set to 1. In the unhedged strategy, ϕ is chosen so that the strategy's delta with respect to the benchmark is 1, whereas in the hedged strategy ϕ is chosen so that its delta is zero. In each strategy, γ is chosen so that the value of the strategy is 1. All the returns in the figure are annualized by setting $\Delta t = 1$.

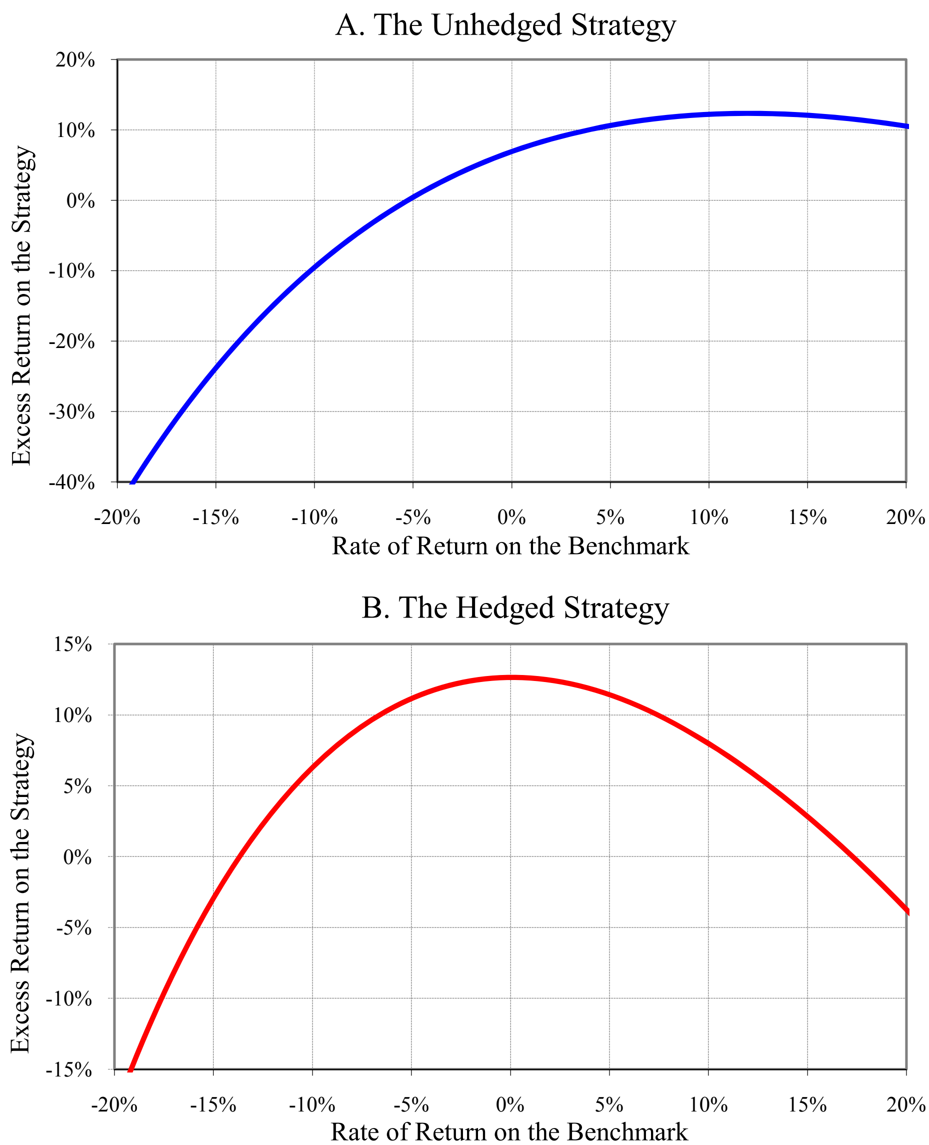

Figure 3: *Options Position which Maximizes the Appraisal Ratio.* For one-dollar investment in the optimal strategy in equation (40), panel A displays the positions in the options of each 1% of moneyness k. Panel B displays the portfolio weight on the options of each 1% of moneyness k. The parameters in the stochastic process of the benchmark are $\mu = 11.43\%$, $\sigma = 14.98\%$ and $r = 5.59\%$ per annum, same as those used in Figure 2.

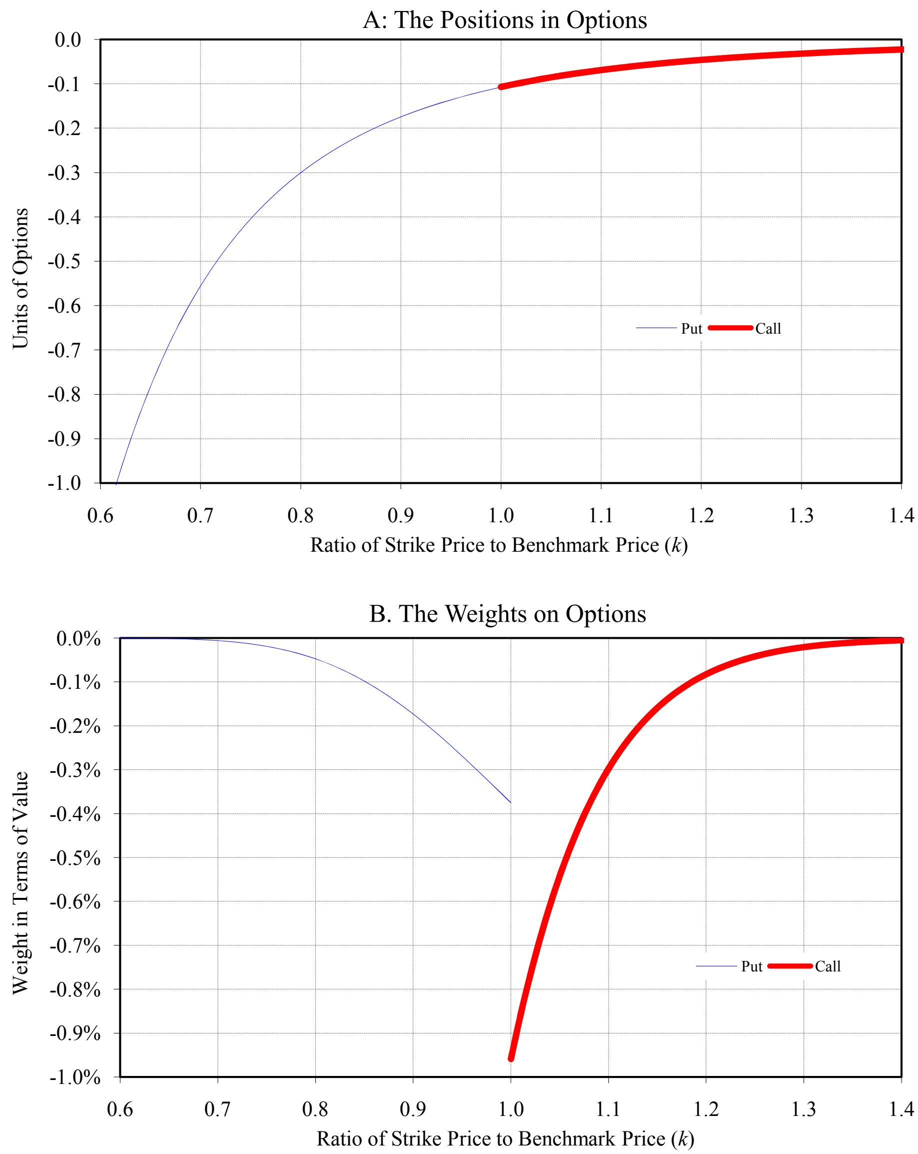

Figure 4: *Delta of the Options in the Optimal Strategy.* Each of the curves from bottom to top is the number of shares in the underlying asset, as a function of the asset price itself, at one month, three months, six months and one year before expiration. Original asset price is normalized to 1.