Explicit Formulation of the Shields Diagram for Incipient Motion of Sediment

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Abstract: The Shields diagram remains the most widely used criterion for incipient motion of sediment. However its implicit nature makes applications rather inconvenient. By deploying Guo's logarithmic matching method twice, this technical note develops an explicit formulation of the Shields diagram, enabling the critical Shields parameter to be determined directly from fluid and sediment characteristics without resorting to any trial and error procedure or iteration. An extended application of the logarithmic matching method is demonstrated.

DOI: 10.1061/(ASCE)0733-9429(2006)132:10(1097)

CE Database subject headings: Sediment transport; Shear stress; Hydraulics; Coastal environment.

Introduction

Sediment transport is a generally important process in fluvial and coastal environments. One of the fundamental aspects of sediment transport is concerned with the critical condition for incipient motion of sediment. To date there have been a large number of experimental studies on this topic, as delineated not only in traditional monographs and textbooks (Graf 1971; Raudkivi 1976; Zhang and Xie 1990; Yang 1996; Chien and Wan 1999; Yalin and da Silva 2001), but also in current papers (Buffington and Montgomery 1997; Buffington 1999; Shvidchenko and Pender 2000).

The critical condition for incipient motion of sediment is normally measured against the critical bed shear stress τ_c. When nondimensionalized by fluid and sediment parameters, it is referred to as the critical Shields parameter $\theta_c \equiv \tau_c/(\rho_s-\rho_w)gd = u_*^2/sgd$, where d=sediment particle diameter; g=gravitational acceleration; $s=\rho_s/\rho_w-1$=submerged specific weight of sediment; u_*=bed shear velocity; and ρ_w and ρ_s=densities of fluid and sediment, respectively. Despite the legendary inconsistencies and misconceptions (Buffington 1999) and experimental discrepancies (Shvidchenko and Pender 2000), the Shields diagram remains the most widely used criterion at present. It establishes a relationship between the critical Shields parameter and the shear Reynolds number $\mathsf{R}_* \equiv du_*/\upsilon$, defined using u_* that is yet to be determined. For a specific set of fluid and sediment parameters, one has to resort to some sort of trial and error procedure or iterations to find the critical bed shear stress. This makes its application in river and coastal engineering rather inconvenient.

A closer scrutiny of the Shields diagram (in a log–log illustration) shows that the critical Shields parameter θ_c follows distinct distributions with the shear Reynolds number R_* (Graf 1971; Raudkivi 1976; Yang 1996; Chien and Wan 1999; Yalin and da Silva 2001). In particular, θ_c declines with increasing R_* following a declining straight line in the lower region as R_* is smaller than around 2, θ_c is constant while R_* is sufficiently large in the upper region (say $\mathsf{R}_*>400$, Graf 1971), and in the intermediate region, the $\theta_c \sim \mathsf{R}_*$ curve follows a saddle shape (Chien and Wan 1999). For the lower and upper regions, the determination of θ_c is quite straightforward with sediment and fluid characteristics, whereas for the intermediate region, it is inconvenient. Yet for the intermediate region, a lower and upper logarithmic asymptote of θ_c in relation to R_* can be identified, and in between there exists a smooth transition. This observation reminds one of the potential application of the recent logarithmic matching method of Guo (2002). This technical note first deploys this method to formulate the relationship between the critical Shields parameter θ_c and the shear Reynolds number R_*, and then an explicit expression between the critical Shields parameter θ_c and the particle Reynolds number $\mathsf{R} \equiv d\sqrt{sgd}/\upsilon$, which is solely determined by fluid and sediment characteristics (here υ is fluid kinematic viscosity). The present explicit formulation allows for expeditious applications of the Shields diagram in the general field of sediment transport.

Formulation of Shields Diagram ($\theta_c - \mathsf{R}_*$)

Guo (2002) proposed the logarithmic matching method to formulate unified relationships for a range of problems in the broad field of hydraulics including sediment transport. The Shields diagram considered herein shows the unique feature in the *intermediate* region that makes the application of the logarithmic matching

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Note. Discussion open until March 1, 2007. Separate discussions must be submitted for individual papers. To extend the closing date by one month, a written request must be filed with the ASCE Managing Editor. The manuscript for this technical note was submitted for review and possible publication on October 7, 2003; approved on October 1, 2006. This technical note is part of the ***Journal of Hydraulic Engineering***, Vol. 132, No. 10, October 1, 2006. ©ASCE, ISSN 0733-9429/2006/10-1097–1099/$25.00.

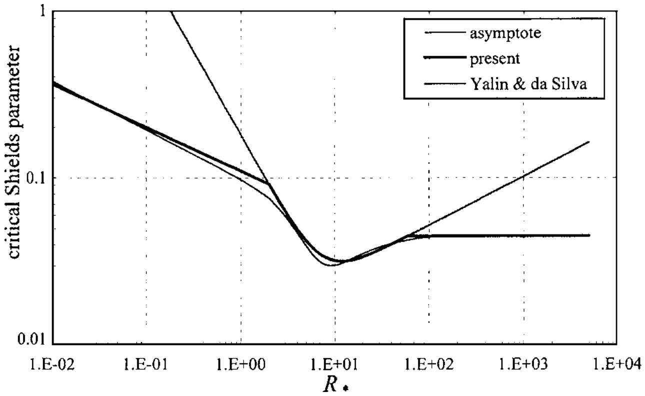

Fig. 1. Present explicit formulation of Shields diagram ($\theta_c \sim \mathsf{R}_*$) as compared with previous relationship (Yalin and da Silva 2001)

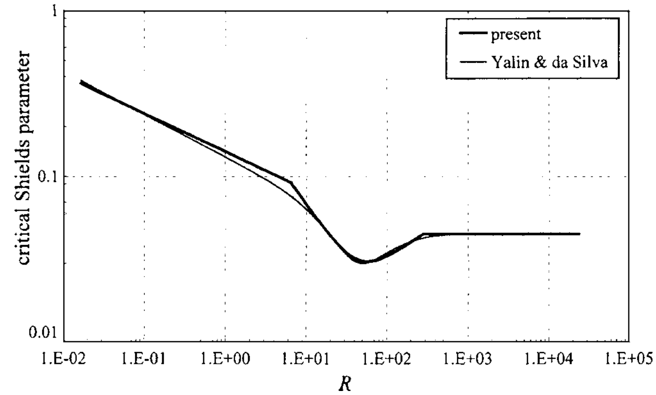

Fig. 2. Critical Shields parameter as function of particle Reynolds number ($\theta_c \sim \mathsf{R}$)

method feasible. In particular, a distinct lower and upper logarithmic asymptote can be readily found, as a smooth transition regime in between (Graf 1971; Yang 1996; Chien and Wan 1999; Yalin and da Silva 2001). The logarithmic matching method has been well described by Guo (2002), and thus the following formulations are directly provided without detailed derivations. According to Guo (2002), there are two possible models that are quite close to each other. Thus only one model is chosen for the following analysis, which read

$$\ln \theta_c = -\ln \mathsf{R}_* + 0.5003 \ln[1 + (0.1359\mathsf{R}_*)^{2.5795}] - 1.7148 \quad (1)$$

where the shear Reynolds number $\mathsf{R}_* \in (2, 60)$.

For the lower region ($\mathsf{R}_* < \approx 2$), the critical Shields parameter θ_c was deemed to decline with increasing shear Reynolds number R_* following a straight line with a slope of −1 (Graf 1971). However, a number of later studies suggest that the slope of the declining straight line should be −0.3 (e.g., Chien and Wan 1999; Yalin and da Silva 2001). In this respect, the explicit formulations of Brownlie (1981) and Vajda (1991) need to be revised because both fit the declining straight line of slope −1. Further, collections of experimental data shown by Yalin and da Silva (2001, Fig. 1.6, p. 8) and also Yang (1996, Fig. 2.3, p. 24) apparently support an even smaller slope (<0.3) of the declining straight line, otherwise the critical Shields parameter could be underestimated compared to the experimental data around $\mathsf{R}_* \in (0.1, 2)$. This observation makes a modification desirable over the explicit relationship of Yalin and da Silva (2001). As such, the following expression for the lower region is suggested

$$\theta_c = 0.1096\mathsf{R}_*^{-0.2607}, \quad \mathsf{R}_* < \approx 2 \quad (2)$$

It is appreciated that the critical Shields parameter approaches a constant when the shear Reynolds number is sufficiently large (say $\mathsf{R}_* > 400$, Graf 1971), and the constant would be 0.045, rather than 0.06 (Chien and Wan 1999; Yalin and da Silva 2001). Numerically, a value of approximately 60 for R_* appears to suffice for defining the upper region (i.e., $\mathsf{R}_* > \approx 60$). Thus one has

$$\theta_c = 0.045 \quad \mathsf{R}_* > \approx 60 \quad (3)$$

A comparison between Eq. (1), along with Eqs. (2) and (3), and that of Yalin and da Silva (2001) is shown in Fig. 1. Fairly good agreement is obtained, apart from the appreciable discrepancy due to the present modified fit to the lower region by a declining straight line of a reduced slope, as stated above. Also shown in Fig. 1 are the lower and upper asymptotes for the logarithmic matching. The lower asymptote essentially represents a declining straight line with a slope of −1 for $\mathsf{R}_* < \approx 2$, as initially thought by Shields (Graf 1971; Chien and Wan 1999).

Critical Shields Parameter versus Particle Reynolds Number ($\theta_c \sim \mathsf{R}$)

For applications, the logarithmic matching formulation Eq. (1) is not yet "well shaped" as the shear Reynolds number R_* involves the unknown bed shear velocity. Nevertheless it has been well known for long that R_* can be represented using the critical Shields parameter θ_c and a nondimensional parameter defined purely with particle and fluid characteristics (e.g., Yalin 1977; Yalin and da Silva 2001). When the particle Reynolds number R is used, one has

$$\mathsf{R}_* = \mathsf{R}\sqrt{\theta_c} \quad (4)$$

While it seems not straightforward to *analytically* eliminate the bed shear velocity from the right-hand side of Eq. (1), it is quite easy to acquire a *discrete* relationship between θ_c and R using Eq. (1). This discrete $\theta_c \sim \mathsf{R}$ relationship is not yet convenient for applications. However it can be readily inferred from Eq. (4) that the $\theta_c \sim \mathsf{R}$ relationship should follow a similar "shape" to the Shields diagram (Fig. 1) except that it is "stretched" in R, compared to R_* (note that $\theta_c < 1$). Alternatively, a lower and upper logarithmic asymptote can be expected as a transition in between. This once again motivates one to deploy Guo's (2002) logarith-

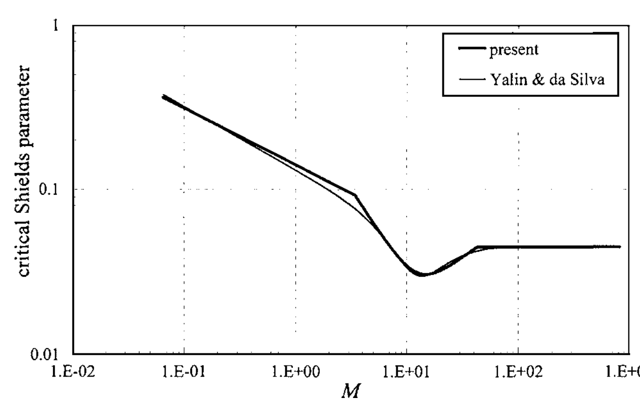

Fig. 3. Critical Shields parameter as function of particle material number ($\theta_c \sim M$)

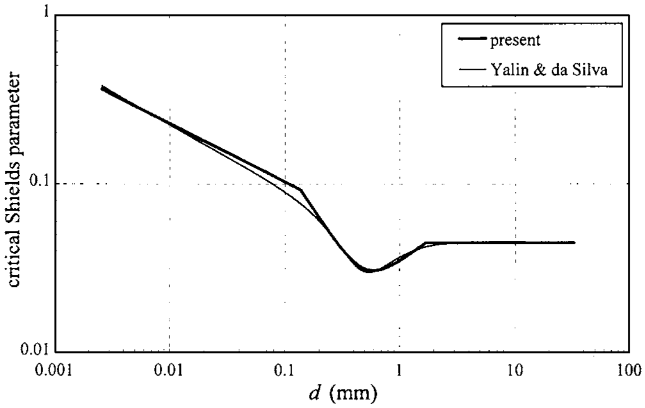

Fig. 4. Critical Shields parameter as function of particle diameter ($\theta_c \sim d$)

mic matching method to formulate an analytical relationship between θ_c and R. In line with Eq. (1), the following explicit formulation is derived:

$$\ln \theta_c = -0.6769 \ln \mathsf{R} + 0.3542 \ln[1 + (0.0223\mathsf{R})^{2.8358}] - 1.1296, \quad \mathsf{R} \in (6.61, 282.84) \qquad (5)$$

Eq. (5) can be rewritten in the power form as

$$\theta_c = \frac{[1 + (0.0223\mathsf{R})^{2.8358}]^{0.3542}}{3.0946\mathsf{R}^{0.6769}}, \quad \mathsf{R} \in (6.61, 282.84) \qquad (6)$$

Using Eqs. (4), (2) and (3), respectively, for the lower and upper regions can be readily translated into the following forms:

$$\theta_c = 0.1414\mathsf{R}^{-0.2306}, \quad \mathsf{R} < \approx 6.61 \qquad (7)$$

$$\theta_c = 0.045, \quad \mathsf{R} > \approx 282.84 \qquad (8)$$

For given values of particle Reynolds number R, the critical Shields parameter can be explicitly calculated with Eqs. (6)–(8). Fig. 2 shows a comparison between the present explicit formulations Eqs. (6)–(8) and the relationship of Yalin and da Silva (2001). A similar illustration is presented in Fig. 3 as the particle Reynolds number R is replaced with the so-called material number $M = \mathsf{R}^{2/3} \equiv d(sg/v^2)^{1/3}$.

For a specific combination of fluid and sediment parameters, the critical Shields parameter can also be illustrated as a function of particle size. In accord with the normally used values of $g = 9.8$ m/s^2, $s = 1.65$, and $v = 1.0$E-6 m^2/s, Fig. 4 shows how the critical Shields parameter varies with particle size according to the Shields diagram.

Conclusion

By deploying the logarithmic matching method of Guo (2002) twice, the Shields diagram for incipient motion of sediment is formulated. The relationship between the critical Shields parameter and the shear Reynolds number is represented by Eqs. (1)–(3). Following this, explicit formulations are provided [Eqs. (6)–(8)], which allow for the determination of the critical Shields parameter using particle Reynolds number or material number that is solely determined by fluid and sediment characteristics. The explicit formulations should find applications in the general area of sediment transport, which are rendered possible by the log-matching method of Guo (2002).

Notation

The following symbols are used in this technical note:

- d = sediment particle diameter;
- g = gravitational acceleration;
- M = particle material number;
- R = particle Reynolds number;
- R_* = shear Reynolds number;
- s = submerged specific gravity of sediment;
- u_* = bed shear velocity;
- v = kinematic viscosity of fluid;
- θ_c = critical Shields parameter for incipient motion of sediment; and
- ρ_w, ρ_s = densities of fluid and sediment, respectively.

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