Practize Kreyzig Homework

CHAPTER ~~19~~ 20. Numerical Methods for Differential Equations

Sec. ~~19~~ 20.1 Methods for First-Order Differential Equations

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3. Euler method. This method is hardly used in practice because it is not accurate enough for most purposes, and there are other methods (Runge-Kutta methods, in particular) that give much more accurate values without too much more work. However, the Euler method explains the principle underlying this class of methods in the simplest possible form, and this is the purpose of the present problem. The latter has the advantage that it concerns a differential equation that can easily be solved exactly, so that you can observe the behavior of the error as the computation is progressing from step to step. The given inital value problem is

$$y' + 5x^4y^2 = 0, \qquad y(0) = 1.$$

For the Euler method you have to write the differential equation in the form

$$y' = f(x, y) = -5x^4y^2. \tag{A}$$

The required step size is $h = 0.2$, so that 10 steps will give approximate solution values from 0 to 2.0. Because of (A) the formula (3) for the Euler method takes the form

$$y_{n+1} = y_n + 0.2(-5x_n^4y_n^2) = y_n - x_n^4y_n^2. \tag{B}$$

Because of the initial condition $y(0) = 1$ your starting values are

$$x = x_0 = 0 \quad \text{and} \quad y = y_0 = 1.$$

The exact solution is obtained by separating variables. Dividing (A) by y^2 on both sides and integrating, you obtain

$$y'/y^2 = -5x^4, \quad -\frac{1}{y} = -x^5 + c.$$

Taking the reciprocal and multiplying by -1 gives

$$y = \frac{1}{x^5 + c^*} \qquad (c^* = -c).$$

From this and the initial condition $y(0) = 1$ you obtain $c^* = 1$. Hence the solution of the problem is

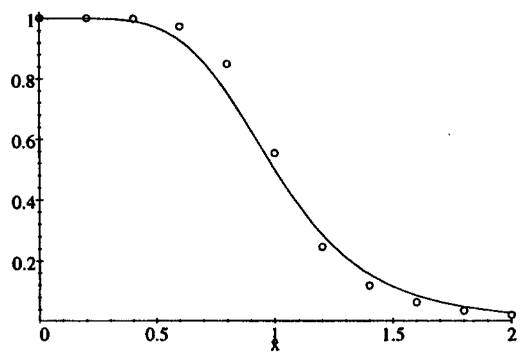
$$y = \frac{1}{x^5 + 1}. \tag{C}$$

Use (C) in computing the error of the approximations obtained from (B). The computations with 10S rounded to 6S give the values shown in the following table.

Table for Problem 3. Computations with Euler's Method

n	x_n	y_n	y_n^2	$x_n^4 y_n^2$	Exact	Error
0	0	1	1	0	1	0
1	0.2	1	1	0.001 600	0.999 680	−0.000 320
2	0.4	0.998 400	0.996 803	0.025 518	0.989 864	−0.008 536
3	0.6	0.972 882	0.946 499	0.122 666	0.927 850	−0.045 031
4	0.8	0.850 216	0.722 867	0.296 086	0.753 194	−0.097 022
5	1.0	0.554 129	0.307 059	0.307 059	0.500 000	−0.054 129
6	1.2	0.247 070	0.061 044	0.126 580	0.286 671	+0.039 601
7	1.4	0.120 490	0.014 518	0.055 772	0.156 783	0.036 293
8	1.6	0.064 718	0.004 188	0.027449	0.087 064	0.022346
9	1.8	0.037 269	0.001 389	0.014 581	0.050 262	0.012 993
10	2.0	0.022 688	0.000 515	0.008 236	0.030 303	0.007 615

It is interesting that the error is neither monotone increasing nor of constant sign, as you might have expected. Of course, this has to do with the particular form of the equation and its solution, which approaches zero as x approaches infinity. The figure shows the behavior of the solution and the approximate values marked as points.



Section 19.1. Problem 3. Solution curve and approximations by Euler's method

13. Classical Runge-Kutta method. This is perhaps the most popular method. The given initial value problem is (see Prob. 11)

$$y' = \frac{2}{x}\sqrt{y - \ln x} + \frac{1}{x}, \qquad y(1) = 0. \tag{A}$$

In Prob. 11 this was solved by Euler's method with $h = 0.1$ for 8 steps from 1.0 to 1.8. The error was determined from the exact solution and was found to increase from 0 to 0.05, approximately. The exact solution can be obtained as follows. The form of the differential equation suggests introducing the new unknown function

$$z = y - \ln x. \quad \text{Then} \quad z' = y' - \frac{1}{x} = \frac{2}{x}\sqrt{z}, \tag{B}$$

where the last equality sign follows by using (A). You can now separate the variables, obtaining

$$\frac{z'}{\sqrt{z}} = \frac{2}{x}.$$

By integration, $2\sqrt{z} = 2\ln x + c^*$, hence $\sqrt{z} = \ln x + c$. Squaring and then using (B), you have

$$z = (\ln x + c)^2, \qquad y = z + \ln x = (\ln x + c)^2 + \ln x.$$

Since $\ln 1 = 0$, you obtain from this and the initial condition $y(1) = c^2 = 0$. Hence the solution is $y = (\ln x)^2 + \ln x$, as shown in Prob. 11. The point of Prob. 13 is a comparison of the accuracy of Euler's method with that of the Runge-Kutta method. Now in the latter you have to compute four auxiliary quantities k_1, k_2, k_3, k_4 per step; hence in the required two steps this amounts to eight such computations, compared to eight steps in the Euler method in Prob. 11; in this sense, the comparison seems fair. The error will turn out to be about half of that of Euler's method. The results of the Runge-Kutta calculations (10S, rounded to 5S) are shown in the table.

Table for Problem 13. Computations with the Runge-Kutta method

x_n	y_n	k_1	k_2	k_3	k_4	Exact	Error
1.0	0	0.4	0.42197	0.44621	0.47501	0	0
1.4	0.43522	0.46529	0.47241	0.47440	0.47436	0.44969	0.01446
1.8	0.90744					0.93328	0.02584

Sec. 19.2 Multistep Methods

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1. Adams-Moulton method. The initial value problem to be solved is

$$y' = f(x,y) = x + y, \quad y(0) = 0. \tag{A}$$

The differential equation is linear. Hence you can solve it exactly, so that no numerical method would be needed. Indeed, write the equation in (A) in the standard form (1), Sec. 1.6,

$$y' - y = x,$$

and solve it by (4), Sec. 1.6, with $p = -1$, hence $h = -x$, obtaining

$$y(x) = e^x\left(\int e^{-x} x\,dx + c\right) = ce^x - x - 1.$$

The initial condition gives $y(0) = c - 0 - 1 = 0$, $c = 1$. Hence the solution of the initial value problem (A) is

$$y(x) = e^x - x - 1. \tag{B}$$

You can later use (B) for determining the errors of the approximate values obtained by the Adams-Moulton method. Now begin with the computation. From (A) you have

$$f_n = f(x_n, y_n) = x_n + y_n, \quad f_{n-1} = f(x_{n-1}, y_{n-1}) = x_{n-1} + y_{n-1}$$

and similarly for the other terms in (7a). Hence (7a) takes the form

$$y^*_{n+1} = y_n + \frac{0.1}{24}[55(x_n + y_n) - 59(x_{n-1} + y_{n-1}) + 37(x_{n-2} + y_{n-2}) - 9(x_{n-3} + y_{n-3})].$$

This gives the predictor. Similarly, the corrector (7b) takes the form

$$y_{n+1} = y_n + \frac{0.1}{24}[9(x_{n+1} + y^*_{n+1}) + 19(x_n + y_n) - 5(x_{n-1} + y_{n-1}) + (x_{n-2} + y_{n-2})].$$

Arrange the numerical values obtained as in Table 19.9 on p. 955.