Inductive Proofs about the Lambda Calculus

Two induction principles

Like before, we have two ways to prove that properties are true of the untyped lambda calculus.

- Structural induction on terms
- Induction on a derivation of $\mathtt{t} \longrightarrow \mathtt{t}'$.

Let's look at an example of each.

Structural induction on terms

To show that a property $\mathcal{P}$ holds for all lambda-terms $\mathtt{t}$, it suffices to show that

- $\mathcal{P}$ holds when $\mathtt{t}$ is a variable;
- $\mathcal{P}$ holds when $\mathtt{t}$ is a lambda-abstraction $\lambda\mathtt{x.}\ \mathtt{t}_1$, assuming that $\mathcal{P}$ holds for the immediate subterm $\mathtt{t}_1$; and
- $\mathcal{P}$ holds when $\mathtt{t}$ is an application $\mathtt{t}_1\ \mathtt{t}_2$, assuming that $\mathcal{P}$ holds for the immediate subterms $\mathtt{t}_1$ and $\mathtt{t}_2$.

Structural induction on terms

To show that a property $\mathcal{P}$ holds for all lambda-terms $\mathtt{t}$, it suffices to show that

- $\mathcal{P}$ holds when $\mathtt{t}$ is a variable;
- $\mathcal{P}$ holds when $\mathtt{t}$ is a lambda-abstraction $\lambda\mathtt{x.}\ \mathtt{t}_1$, assuming that $\mathcal{P}$ holds for the immediate subterm $\mathtt{t}_1$; and
- $\mathcal{P}$ holds when $\mathtt{t}$ is an application $\mathtt{t}_1\ \mathtt{t}_2$, assuming that $\mathcal{P}$ holds for the immediate subterms $\mathtt{t}_1$ and $\mathtt{t}_2$.

N.b.: The variant of this principle where "immediate subterm" is replaced by "arbitrary subterm" is also valid. (Cf. *ordinary induction* vs. *complete induction* on the natural numbers.)

An example of structural induction on terms

Define the set of *free variables* in a lambda-term as follows:

$$
\begin{aligned}
&FV(\mathtt{x}) = \{\mathtt{x}\}\\
&FV(\lambda\mathtt{x}.\mathtt{t}_1) = FV(\mathtt{t}_1) \setminus \{\mathtt{x}\}\\
&FV(\mathtt{t}_1\ \ \mathtt{t}_2) = FV(\mathtt{t}_1) \cup FV(\mathtt{t}_2)
\end{aligned}
$$

Define the *size* of a lambda-term as follows:

$$
\begin{aligned}
&size(\mathtt{x}) = 1\\
&size(\lambda\mathtt{x}.\mathtt{t}_1) = size(\mathtt{t}_1) + 1\\
&size(\mathtt{t}_1\ \ \mathtt{t}_2) = size(\mathtt{t}_1) + size(\mathtt{t}_2) + 1
\end{aligned}
$$

Theorem: $|FV(\mathtt{t})| \leq size(\mathtt{t})$.

An example of structural induction on terms

Theorem: $|FV(\mathtt{t})| \leq size(\mathtt{t})$.

Proof: By induction on the structure of $\mathtt{t}$.

- If $\mathtt{t}$ is a variable, then $|FV(\mathtt{t})| = 1 = size(\mathtt{t})$.
- If $\mathtt{t}$ is an abstraction $\lambda\mathtt{x}.\ \mathtt{t}_1$, then

$$\begin{array}{lll} & |FV(\mathtt{t})| & \\ = & |FV(\mathtt{t}_1) \setminus \{\mathtt{x}\}| & \text{by defn} \\ \leq & |FV(\mathtt{t}_1)| & \text{by arithmetic} \\ \leq & size(\mathtt{t}_1) & \text{by induction hypothesis} \\ \leq & size(\mathtt{t}_1) + 1 & \text{by arithmetic} \\ = & size(\mathtt{t}) & \text{by defn.} \end{array}$$

An example of structural induction on terms

Theorem: $|FV(\mathtt{t})| \leq size(\mathtt{t})$.

Proof: By induction on the structure of $\mathtt{t}$.

- If $\mathtt{t}$ is an application $\mathtt{t}_1\ \mathtt{t}_2$, then

$$
\begin{array}{cll}
 & |FV(\mathtt{t})| & \\
= & |FV(\mathtt{t}_1) \cup FV(\mathtt{t}_2)| & \text{by defn} \\
\leq & max(|FV(\mathtt{t}_1)|, |FV(\mathtt{t}_2)|) & \text{by arithmetic} \\
\leq & max(|size(\mathtt{t}_1)|, |size(\mathtt{t}_2)|) & \text{by IH and arithmetic} \\
\leq & |size(\mathtt{t}_1)| + |size(\mathtt{t}_2)| & \text{by arithmetic} \\
\leq & |size(\mathtt{t}_1)| + |size(\mathtt{t}_2)| + 1 & \text{by arithmetic} \\
= & size(\mathtt{t}) & \text{by defn.}
\end{array}
$$

Induction on derivations

Recall that the reduction relation is defined as the smallest binary relation on terms satisfying the following rules:

$$(\lambda \mathtt{x}.\mathtt{t}_{12})\ \mathtt{v}_2 \longrightarrow [\mathtt{x} \mapsto \mathtt{v}_2]\mathtt{t}_{12} \qquad \text{(E-AppAbs)}$$

$$\frac{\mathtt{t}_1 \longrightarrow \mathtt{t}_1'}{\mathtt{t}_1\ \mathtt{t}_2 \longrightarrow \mathtt{t}_1'\ \mathtt{t}_2} \qquad \text{(E-App1)}$$

$$\frac{\mathtt{t}_2 \longrightarrow \mathtt{t}_2'}{\mathtt{v}_1\ \mathtt{t}_2 \longrightarrow \mathtt{v}_1\ \mathtt{t}_2'} \qquad \text{(E-App2)}$$

Induction on derivations

Induction principle for the small-step evaluation relation.

To show that a property $\mathcal{P}$ holds for all derivations of $\mathtt{t} \longrightarrow \mathtt{t}'$, it suffices to show that

- $\mathcal{P}$ holds for all derivations that use the rule E-AppAbs;
- $\mathcal{P}$ holds for all derivations that end with a use of E-App1 assuming that $\mathcal{P}$ holds for all subderivations; and
- $\mathcal{P}$ holds for all derivations that end with a use of E-App2 assuming that $\mathcal{P}$ holds for all subderivations.

Example

Theorem: if $\mathtt{t} \longrightarrow \mathtt{t}'$ then $FV(\mathtt{t}) \supseteq FV(\mathtt{t}')$.

Induction on derivations

We must prove, for all derivations of $\texttt{t} \longrightarrow \texttt{t}'$, that $FV(\texttt{t}) \supseteq FV(\texttt{t}')$.

There are three cases.

Induction on derivations

We must prove, for all derivations of $\mathtt{t} \longrightarrow \mathtt{t}'$, that $FV(\mathtt{t}) \supseteq FV(\mathtt{t}')$.

There are three cases.

- If the derivation of $\mathtt{t} \longrightarrow \mathtt{t}'$ is just a use of E-AppAbs, then $\mathtt{t}$ is $(\lambda\mathtt{x}.\mathtt{t}_1)\mathtt{v}$ and $\mathtt{t}'$ is $[\mathtt{x}\mapsto\mathtt{v}]\mathtt{t}_1$. Reason as follows:

$$\begin{aligned} FV(\mathtt{t}) &= FV((\lambda\mathtt{x}.\mathtt{t}_1)\mathtt{v}) \\ &= FV(\mathtt{t}_1)/\{\mathtt{x}\} \cup FV(\mathtt{v}) \\ &\supseteq FV([\mathtt{x}\mapsto\mathtt{v}]\mathtt{t}_1) \\ &= FV(\mathtt{t}') \end{aligned}$$

- If the derivation ends with a use of E-App1, then $\mathtt{t}$ has the form $\mathtt{t}_1\ \mathtt{t}_2$ and $\mathtt{t}'$ has the form $\mathtt{t}_1'\ \mathtt{t}_2$, and we have a subderivation of $\mathtt{t}_1 \longrightarrow \mathtt{t}_1'$

  By the induction hypothesis, $FV(\mathtt{t}_1) \supseteq FV(\mathtt{t}_1')$. Now calculate:

$$\begin{aligned} FV(t) &= FV(\mathtt{t}_1\ \mathtt{t}_2) \\ &= FV(\mathtt{t}_1) \cup FV(\mathtt{t}_2) \\ &\supseteq FV(\mathtt{t}_1') \cup FV(\mathtt{t}_2) \\ &= FV(\mathtt{t}_1'\ \mathtt{t}_2) \\ &= FV(t') \end{aligned}$$

- If the derivation ends with a use of E-App1, then $\mathtt{t}$ has the form $\mathtt{t}_1\ \mathtt{t}_2$ and $\mathtt{t}'$ has the form $\mathtt{t}_1'\ \mathtt{t}_2$, and we have a subderivation of $\mathtt{t}_1 \longrightarrow \mathtt{t}_1'$

  By the induction hypothesis, $FV(\mathtt{t}_1) \supseteq FV(\mathtt{t}_1')$. Now calculate:

$$\begin{aligned} FV(t) &= FV(\mathtt{t}_1\ \mathtt{t}_2) \\ &= FV(\mathtt{t}_1) \cup FV(\mathtt{t}_2) \\ &\supseteq FV(\mathtt{t}_1') \cup FV(\mathtt{t}_2) \\ &= FV(\mathtt{t}_1'\ \mathtt{t}_2) \\ &= FV(t') \end{aligned}$$

- If the derivation ends with a use of E-App2, the argument is similar to the previous case.