MI46B - TAREA 1

Un medio sujeto a fuerzas externas en el origen de un sistema (x,y). Los desplazamientos inducidos en el cuerpo vienen dados por:

$$u_x = \frac{1}{2G}\left(\frac{xy}{r^2} + C_1\right)$$
$$u_y = \frac{1}{2G}\left[\frac{y^2}{r^2} - (3-4v)lnr + C_2\right]$$
$$r^2 = x^2 + y^2$$

(a) Definir las expresiones para los componentes de deformación

$$\varepsilon_{xx} = -\frac{\partial u_x}{\partial x}$$
$$\varepsilon_{xx} = -\frac{\partial}{\partial x}\left[\frac{1}{2G}\left(\frac{xy}{r^2} + C_1\right)\right]$$
$$\varepsilon_{xx} = -\frac{1}{2G}\left[\frac{\partial}{\partial x}\left(\frac{xy}{r^2} + C_1\right)\right]$$
$$\varepsilon_{xx} = -\frac{1}{2G}\frac{\partial}{\partial x}\left(\frac{xy}{x^2+y^2}\right)$$
$$\varepsilon_{xx} = -\frac{1}{2G}\left[\frac{y(x^2+y^2) - xy2x}{(x^2+y^2)^2}\right]$$
$$\varepsilon_{xx} = -\frac{1}{2G}\left[\frac{yr^2 - 2x^2y}{r^4}\right]$$
$$\Rightarrow \boldsymbol{\varepsilon_{xx}} = \boldsymbol{\frac{1}{2G}\left[-\frac{y}{r^2} + \frac{2x^2y}{r^4}\right]}$$

$$\varepsilon_{yy} = -\frac{\partial u_y}{\partial y}$$
$$\varepsilon_{yy} = -\frac{\partial}{\partial y}\left\{\frac{1}{2G}\left[\frac{y^2}{r^2} - (3-4\upsilon)lnr + C_2\right]\right\}$$
$$\varepsilon_{yy} = -\frac{1}{2G}\left\{\frac{\partial}{\partial y}\left[\frac{y^2}{x^2+y^2} - (3-4\upsilon)ln\sqrt{x^2+y^2} + C_2\right]\right\}$$
$$\varepsilon_{yy} = -\frac{1}{2G}\left[\frac{2y(x^2+y^2) - y^2 2y}{(x^2+y^2)^2} - \frac{(3-4v)}{\sqrt{(x^2+y^2)}}\frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}2y\right]$$
$$\varepsilon_{yy} = -\frac{1}{2G}\left[\frac{2y}{r^2} - \frac{2y^3}{r^4} - \frac{(3-4v)y}{r^2}\right]$$
$$\varepsilon_{yy} = -\frac{1}{2G}\left[-\frac{2y^3}{r^4} + \frac{y}{r^2}\left(2-(3-4v)\right)\right]$$

$$\varepsilon_{yy} = -\frac{1}{2G}\left[-\frac{2y^3}{r^4} + \frac{y}{r^2}(-1+4\upsilon)\right]$$

$$\Rightarrow \boldsymbol{\varepsilon_{yy}} = \frac{\mathbf{1}}{\mathbf{2G}}\left[\frac{\mathbf{2y^3}}{\mathbf{r^4}} + \frac{\boldsymbol{y}}{\boldsymbol{r^2}}(\mathbf{1}-\mathbf{4}\boldsymbol{\upsilon})\right]$$

$$\gamma_{xy} = -\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)$$

$$\frac{\partial u_y}{\partial x} = \frac{\partial}{\partial x}\left\{\frac{1}{2G}\left[\frac{y^2}{r^2} - (3-4\upsilon)lnr + C_2\right]\right\}$$

$$\frac{\partial u_y}{\partial x} = \frac{1}{2G}\left\{\frac{\partial}{\partial y}\left[\frac{y^2}{x^2+y^2} - (3-4\upsilon)ln\sqrt{x^2+y^2} + C_2\right]\right\}$$

$$\frac{\partial u_y}{\partial x} = \frac{1}{2G}\left[-\frac{y^2 2x}{(x^2+y^2)^2} - (3-4\upsilon)\frac{1}{\sqrt{x^2+y^2}}\frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}2x\right]$$

$$\frac{\partial u_y}{\partial x} = \frac{1}{2G}\left[-\frac{2xy^2}{r^4} - \frac{(3-4\upsilon)}{r^2}x\right]$$

$$\frac{\partial u_x}{\partial y} = \frac{\partial}{\partial y}\left[\frac{1}{2G}\left(\frac{xy}{r^2} + C_1\right)\right]$$

$$\frac{\partial u_x}{\partial y} = \frac{1}{2G}\left[\frac{\partial}{\partial y}\left(\frac{xy}{x^2+y^2} + C_1\right)\right]$$

$$\frac{\partial u_x}{\partial y} = \frac{1}{2G}\left[\frac{x(x^2+y^2) - xy\cdot 2y}{(x^2+y^2)^2}\right]$$

$$\frac{\partial u_x}{\partial y} = \frac{1}{2G}\left(\frac{xr^2}{r^4} - \frac{2xy^2}{r^4}\right)$$

$$\frac{\partial u_x}{\partial y} = \frac{1}{2G}\left(\frac{x}{r^2} - \frac{2xy^2}{r^4}\right)$$

$$\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} = \frac{1}{2G}\left[-\frac{4xy^2}{r^4} + \frac{x}{r^2} - \frac{(3-4\upsilon)}{r^2}\,x\right]$$

$$\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} = \frac{1}{2G}\left[-\frac{4xy^2}{r^4} - (1-2\upsilon)\frac{2x}{r^2}\right]$$

$$\Rightarrow \boldsymbol{\gamma_{xy}} = \frac{\mathbf{1}}{\mathbf{2G}}\left[\frac{\mathbf{4xy^2}}{\mathbf{r^4}} + (\mathbf{1}-\mathbf{2}\boldsymbol{\upsilon})\frac{\mathbf{2x}}{\mathbf{r^2}}\right]$$

(b) Verificar que la solución cumple las ecuaciones de compatibilidad

Las ecuaciones de compatibilidad son las siguientes:

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

$$2\frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

Cada una de las anteriores ecuaciones con sus respectivas combinaciones. En este caso solamente se debe verificar la primera ecuación antes escrita.

$$\frac{\partial \varepsilon_{xx}}{\partial y} = \frac{1}{2G}\left[-\frac{1}{x^2+y^2} + \frac{8y^3}{(x^2+y^2)^2} + \frac{2x^2}{(x^2+y^2)^2} - \frac{8x^2y^2}{(x^2+y^2)^3}\right]$$

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} = \frac{1}{2G}\left[\frac{6y}{(x^2+y^2)} - \frac{8y^3}{(x^2+y^2)^3} - \frac{24x^2y}{(x^2+y^2)^3} + \frac{48x^2y^3}{(x^2+y^2)^4}\right]$$

$$\frac{\partial \varepsilon_{yy}}{\partial x} = \frac{1}{2G}\left[-\frac{8y^3x}{(x^2+y^2)^3} - \frac{2y(1-4\upsilon)x}{(x^2+y^2)^2}\right]$$

$$\frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{1}{2G}\left[\frac{48x^2y^3}{(x^2+y^2)^4} - \frac{8y^3}{(x^2+y^2)^3} + \frac{8y(1-4\upsilon)x^2}{(x^2+y^2)^3} - \frac{2y(1-4\upsilon)}{(x^2+y^2)^2}\right]$$

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{1}{2G}\left[\left(\frac{6y}{(x^2+y^2)} - \frac{8y^3}{(x^2+y^2)^3} - \frac{24x^2y}{(x^2+y^2)^3} + \frac{48x^2y^3}{(x^2+y^2)^4}\right) + \left(\frac{48x^2y^3}{(x^2+y^2)^4} - \frac{8y^3}{(x^2+y^2)^3} + \frac{8y(1-4\upsilon)x^2}{(x^2+y^2)^3} - \frac{2y(1-4\upsilon)}{(x^2+y^2)^2}\right)\right]$$

$$\Rightarrow \boldsymbol{\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = -\frac{2y}{(x^2+y^2)^4\,G}\left(3x^4 - 18x^2y^2 + 3y^4 + 6x^4\upsilon + 4x^2\upsilon y^2 - 2\upsilon y^4\right)}$$

$$\frac{\partial \gamma_{xy}}{\partial y} = \frac{1}{2G}\left[\frac{8xy}{(x^2+y^2)^2} - \frac{16y^3x}{(x^2+y^2)^3} - \frac{2(2-4\upsilon)xy}{(x^2+y^2)^2}\right]$$

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{1}{2G}\left[\frac{8y}{(x^2+y^2)^2} - \frac{32x^2y}{(x^2+y^2)^3} + \frac{96x^2y^3}{(x^2+y^2)^4} - \frac{16y^3}{(x^2+y^2)^3} - \frac{2(2-4\upsilon)y}{(x^2+y^2)^2} + \frac{8(2-2\upsilon)x^2y}{(x^2+y^2)^3}\right]$$

$$\Rightarrow \boldsymbol{\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = -\frac{2y}{(x^2+y^2)^4\,G}\left(3x^4 - 18x^2y^2 + 3y^4 + 6x^4\upsilon + 4x^2\upsilon y^2 - 2\upsilon y^4\right)}$$

$$\Rightarrow \boldsymbol{\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}}$$

(c) Determinar una expresión para los esfuerzos en el caso isótropo

Los esfuerzos se pueden determinar con la siguiente ecuación:

$$\begin{bmatrix} \sigma xx \\ \sigma yy \\ \sigma zz \\ \sigma xy \\ \sigma yz \\ \sigma zx \end{bmatrix} = \frac{E(1-v)}{(1+v)(1-2v)} \begin{bmatrix} 1 & v/(1-v) & v/(1-v) & 0 & 0 & 0 \\ v/(1-v) & 1 & v/(1-v) & 0 & 0 & 0 \\ v/(1-v) & v/(1-v) & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-2v)/(2(1-v)) & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-2v)/(2(1-v)) & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-2v)/(2(1-v)) \end{bmatrix} \begin{bmatrix} \varepsilon xx \\ \varepsilon yy \\ \varepsilon zz \\ \gamma xy \\ \gamma yz \\ \gamma zx \end{bmatrix}$$

$$\begin{bmatrix} \sigma xx \\ \sigma yy \\ \sigma zz \\ \sigma xy \\ \sigma yz \\ \sigma zx \end{bmatrix} = \frac{E(1-v)}{(1+v)(1-2v)} \begin{bmatrix} 1 & v/(1-v) & v/(1-v) & 0 & 0 & 0 \\ v/(1-v) & 1 & v/(1-v) & 0 & 0 & 0 \\ v/(1-v) & v/(1-v) & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-2v)/(2(1-v)) & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-2v)/(2(1-v)) & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-2v)/(2(1-v)) \end{bmatrix} \begin{bmatrix} \varepsilon xx \\ \varepsilon yy \\ 0 \\ \gamma xy \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \sigma xx \\ \sigma yy \\ \sigma zz \\ \sigma xy \\ \sigma yz \\ \sigma zx \end{bmatrix} = \frac{E(1-v)}{(1+v)(1-2v)} \begin{bmatrix} \varepsilon xx + (\varepsilon yy\ v)/(1-v) \\ (\varepsilon xx\ v)/(1-v) + \varepsilon yy \\ (\varepsilon xx\ v)/(1-v) + (\varepsilon yy\ v)/(1-v) \\ \gamma xy\ (1-2v)/(2(1-v)) \\ 0 \\ 0 \end{bmatrix}$$

De lo anterior y reemplazando los valores, se obtienen las siguientes expresiones para los esfuerzos:

$$\sigma_{xx} = \frac{E}{2G}\left[\frac{y(2\nu r^2 + x^2 - y^2}{r^4(1-\nu)}\right]$$

$$\sigma_{yy} = -\frac{E}{2G}\left[\frac{y(2\nu r^2 - 3y^2 - x^2}{r^4(1-\nu)}\right]$$

$$\sigma_{zz} = \frac{E}{G}\frac{\nu y}{r^2(1+\nu)}$$

$$\sigma_{xy} = -\frac{E}{2G}\left[\frac{x(2\nu r^2 - 3y^2 - x^2}{r^4(1+\nu)}\right]$$

$$\sigma_{yz} = 0$$

$$\sigma_{zx} = 0$$