OPTIMAL REPLACEMENT IN THE PROPORTIONAL HAZARDS MODEL*

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ABSTRACT

In this paper, we examine a replacement problem for a system subject to stochastic deterioration. Upon failure the system must be replaced by a new one and a failure cost is incurred. If the system is replaced before failure a smaller cost is incurred. The failure of the system depends both on its age and also on values of a diagnostic stochastic process observable at discrete points of time. Cox's proportional hazards model is used to describe the failure rate of the system. We consider the problem of specifying a replacement rule which minimizes the long-run expected average cost per unit time. The form of the optimal replacement policy is found and an algorithm based on a recursive computational procedure is presented which can be used to obtain the optimal policy and the optimal expected average cost.

Keywords: Replacement policy, dynamic programming, optimization, proportional hazards modelling.

RÉSUMÉ

Dans cet article on considère un problème de remplacement pour un système sujet à une détérioration aléatoire. Lorsque le système tombe en panne, il doit être remplacé par un nouveau système et on encourt alors un coût de défaillance. Si le système est remplacé avant de tomber en panne, un coût moins élevé est encouru. Les pannes du système dépendent de son âge et de valeurs prises par un processus stochastique de diagnostic observable de façon ponctuelle dans le temps. On utilise le modèle des taux de défaillance proportionnels (*"proportional hazards model"*) de Cox pour décrire le taux de panne du système. On considère le problème de la détermination d'une politique de remplacement qui minimise l'espérance mathématique du coût moyen par unité de temps sur une longue période. La forme de la politique optimale est obtenue et l'on présente un algorithme basé sur un processus de calcul récursif dont on peut se servir pour calculer la politique optimale et l'espérance du coût moyen optimal.

Mots-clés : politique de remplacement, programmation dynamique, optimisation, modélisation avec taux de défaillance proportionnels.

1. INTRODUCTION

We consider a system that is subject to failure. The failure rate of the system is a function of age but can also depend on the values of concomitant variables describing the effect of the environment in which it operates.

In engineering applications usually additional concomitant information is available such as that obtained through SOAP (Spectrometric Oil Analysis Program). This information should then be taken into consideration by a decision maker, who must decide when to suspend operations to perform preventive maintenance.

To model the effect of concomitant variables on failure time, we consider the proportional hazards model which has been widely used in medical research but only recently applied to engineering reliability problems (*e.g.* Bendell (1985), and Jardine *et al.* (1987)). In the PHM, it is assumed that the

failure rate of a system is the product of a baseline failure rate $h_0(\cdot)$ dependent on the age of the system and a positive function $\psi(\cdot)$ dependent only on the values of concomitant variables.

We study the problem of optimal replacement in the PHM, *i.e.*, the problem of specifying a replacement rule which minimizes the long-run expected average cost per unit time. Optimal replacement problems have been studied by many researchers. Considerable attention has been paid to optimal replacement problems in shock models with additive damage (*e.g.* Taylor (1975), Gottlieb (1982), Posner and Zuckerman (1986)). The form of optimal replacement policy was found

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and conditions were determined under which a control limit policy is optimal. Bergman (1978) studied an optimal replacement problem with a nondecreasing damage process X. He showed that the optimal replacement policy is a control limit policy with respect to the process X, *i.e.*, a replacement is

performed either at failure or when X first reaches or exceeds a given level ξ. A more detailed survey can be found in Valdez-Flores and Feldman (1989).

In our model, the failure of the system depends both on the age of the system and on the values of a diagnostic stochastic process Z. We assume that the values of process Z are known only at discrete time points and decisions are made only at these points, which is the case in most real situations. The classical cost function is assumed. Thus each replacement costs C, while each failure costs $C+K, C>0, K>0$. The problem is to find the replacement policy which minimizes the expected average cost.

Two variants of the model are examined. In variant 1, the system can be preventively replaced at any time (but the decision has to be made at a decision instant), while in variant 2 the system is available for preventive replacement only at the decision instants. In both variants, it is assumed that the system deteriorates continuously over time and can fail at any instant. This distinguishes our model from the shock models with additive damage in which failures can occur only at times of shocks (see *e.g.* Posner and Zuckerman (1986) and references in their paper) and also from the models with continuous observations of the damage process. The transition times in our model are random variables dependent on the present state and action taken so that the decision processes $\{X_n, a_n, \tau_n\}$, where X_n is the n^{th} state, a_n is the n^{th} action chosen and τ_n is the time between the $(n-1)^{\text{th}}$ and the n^{th} transition, are semi-Markov decision processes for both variants (see *e.g.* Ross (1970), p.156).

We derive the average-cost optimality equations, examine the structure of stationary optimal policies and propose an algorithm for finding optimal replacement policies for both variants.

2. DEVELOPMENT OF THE MODEL

Let $Z=\{Z_t, t\geq 0\}$ be a right continuous stochastic process with state space $R^+=[0,+\infty)$ that can influence the time to failure of the equipment. We assume that the failure rate is the product of a baseline failure rate dependent only on the age of the unit and a positive function $\psi(\cdot)$ dependent on the values of the stochastic process Z. The process Z is a diagnostic (damage) process that reflects the effect of the operating environment on the system (*e.g.* Z can be the level of metal particle in engine oil). Thus the failure rate at time t can be expressed as

$$h(t, Z_t) = h_0(t)\psi(Z_t) \text{ for } t\geq 0 \tag{1}$$

and the survivor function is given by

$$P(T>t|Z_s,\ 0\leq s\leq t) = \exp\left(-\int_0^t h_0(s)\psi(Z_s)\,ds\right),\ t\geq 0 \tag{2}$$

where T is the time to failure of the system (*e.g.* Cox and Oakes (1984)).

In most real situations the values of Z are usually known only in some discrete points of time $t_0, t_1, t_2, \ldots$ Then we can approximate the stochastic process $\{Z_t, t\geq 0\}$ by the right continuous jump process $\{Z_t^*, t\geq 0\}$ which increases and decreases by jumps at times $t_0, t_1, \ldots$, otherwise is constant.

In our model, we assume that concomitant information is available at time points $0, \Delta, 2\Delta, \ldots, \Delta>0$ in a given replacement cycle and let Z_k be the value of concomitant variable at time $k\Delta$ after the last replacement, $Z_0=0$ and let

$$P(Z_{k+1}\leq y|Z_1,\ldots,Z_k) = G(y|k, Z_k) \tag{3}$$

where $G(y|k,z)$ is the conditional distribution function measurable in all arguments. From (2), we have for $t \in [0, \Delta]$

$$P(T > k\Delta + t | T > k\Delta, Z_1, \ldots, Z_k) = \exp\left(-\psi(Z_k) \int_{k\Delta}^{k\Delta+t} h_0(s)ds\right)$$
$$\equiv R(k, Z_k, t). \tag{4}$$

Our objective is to find the replacement policy which minimizes the long-run expected average cost per unit time. For both variants, the decisions can be made at times $n\Delta$, $n = 0, 1, 2, \ldots$ in a given replacement cycle and the system is in state (k, z) if the age of the unit is $k\Delta$ and the present value of concomitant variable is z. Thus the state space

$$S = N \times R^+, \text{ where } N = \{0, 1, 2, \ldots\}, R^+ = [0, +\infty). \tag{5}$$

For variant 1, the action space includes three kinds of actions: (i) planned replacement after a time units ($0 < a < \Delta$), (ii) no replacement ($a = +\infty$), and (iii) an immediate replacement followed by a future replacement after a time units (this action is denoted by the doubleton $\langle 0, a \rangle$).

Therefore, the action space is

$$A = B \cup \{\langle 0, a \rangle, a \in B\}, \text{ where } B = (0, \Delta) \cup \{+\infty\}. \tag{6}$$

The reason why we consider actions $\langle 0, a \rangle$ is that if an immediate replacement takes place in state x and no other action is taken (say action 0) then the next state is 0 and the sojourn time in state x is equal to zero. However, we will show in Section 3, that the replacement problem for variant 1 is equivalent to the problem of solving an optimality equation with action space $A' = [0, \Delta) \cup \{+\infty\}$.

For variant 2, there are only 2 possible actions $\{0, +\infty\}$ at each decision instant where 0 corresponds to an immediate replacement and $+\infty$ corresponds to non-replacement.

Denote

$$F(t|k, z) = 1 - R(k, z, t) \text{ for } t \geq 0. \tag{7}$$

If we take an action $a \in B$ in state (k, z) then, for variant 1, the mean sojourn time is

$$\tau(k, z, a) = \int_0^a tF(dt|k, z) + aR(k, z, a) = \int_0^a R(k, z, t)\, dt,$$

$$\tau(k, z, +\infty) = \int_0^\Delta R(k, z, t)\, dt = \tau(k, z, \Delta)$$

and

$$\tau(k, z, \langle 0, a \rangle) = \tau(0, 0, a), \quad \tau(k, z, \langle 0, +\infty \rangle) = \tau(0, 0, +\infty). \tag{8}$$

For variant 2,

$$\tau_2(k, z, +\infty) = \tau(k, z, +\infty), \quad \tau_2(k, z, 0) = \tau(0, 0, +\infty). \tag{8'}$$

In the next section, we derive the average cost optimality equations for both variants under certain assumptions and show that optimal replacement policies can be found in the class of stationary policies.

3. DERIVATION OF THE OPTIMALITY EQUATIONS

In this section, we introduce the following assumptions.

1. In order to insure that the transitions do not take place too quickly, we assume that there exist $\sigma > 0$, $\eta > 0$ such that the conditional probability of survival

$$R(k, z, \sigma) \geq \eta \text{ for } (k, z) \in S.$$

2. $R(k,z,\Delta) \leq \alpha < 1$, $(k,z) \in S$, which means that in each state the probability of failure in the next period is positive.
3. The state of the process Z is stochastically increasing in (k,z), *i.e.*, the distribution function $G(\cdot|k,z)$ is a decreasing function in (k,z).
4. $h_0(t)$ is nondecreasing, *i.e.*, the system deteriorates with age.
5. $\psi(z)$ is nondecreasing, $\psi(0) = 1$, which means that the hazard rate of the system is a nondecreasing function of the state of the process Z.

In the following theorem, we derive the average cost optimality equations for both variants.

Theorem 1: *Let assumptions 1-5 be satisfied. Then, for variant i, i = 1,2, there exist a bounded nondecreasing measurable function v^i defined on S with values in R^+ and a constant $g^i \geq 0$ such that*

$$v^1(k,z) = \min\{\inf_{a\in[0,\Delta)}\{V(k,z,a,g^1,v^1)\},\ W(k,z,+\infty,g^1,v^1)\}, \tag{9}$$

$$v^2(k,z) = \min\{C + v^2(0,0),\ W(k,z,+\infty,g^2,v^2)\} \tag{9'}$$

where

$$\begin{aligned} V(k,z,a,g,v) &= (K + C + v(0,0))(1 - R(k,z,a)) + (C + v(0,0))R(k,z,a) \\ &\quad - g\tau(k,z,a) \quad \text{for } a \in (0,\Delta), \end{aligned} \tag{10}$$

$$\begin{aligned} W(k,z,+\infty,g,v) &= [K + C + v(0,0)][1 - R(k,z,\Delta)] + \int v(k+1,y)G(dy|k,z) \\ &\quad \cdot R(k,z,\Delta) - g\tau(k,z,+\infty) \end{aligned} \tag{11}$$

for $(k,z) \in S$.

First, we prove the monotonicity of functions V and W.

Lemma 1: *Let assumptions 3-5 be satisfied. Then functions $V(k,z,a,g,v)$ and $W(k,z,+\infty,g,v)$ defined by (10) and (11) are nondecreasing in (k,z) for any nonnegative, nondecreasing function v such that $v(k,z) \leq K + C + v(0,0)$ for $(k,z) \in S$ and for any positive constant g.*

Proof: It follows from assumptions 4 and 5 that functions $R(k,z,a)$ and $\tau(k,z,a)$ defined by (4) and (8) are nonincreasing in (k,z) for any $a \in A$. Next, since $v(k,z)$ is nondecreasing in (k,z), it follows from assumption 3, that

$$\int [v(k+1,y) - K - C - v(0,0)]G(dy|k,z)R(k,z,\Delta)$$

is nondecreasing in (k,z) (*e.g.* Ross (1983), p. 154), so that $W(k,z,+\infty,g,v)$ is nondecreasing for any $g > 0$. For function V, we have

$$V(k,z,a,g,v) = K + C + v(0,0) - KR(k,z,a) - g\tau(k,z,a)$$

and the result follows, since R and τ are nonincreasing in (k,z). This completes the proof. •

Proof of Theorem 1: First, we examine variant 1. Since for $a \to 0$, $\tau(k,z,a) \to 0$, we first establish the optimality equation for restricted action space A_ϵ, where

$$A_\epsilon = B_\epsilon \cup \{\langle 0,a\rangle, a \in B_\epsilon\}, \quad B_\epsilon = [\epsilon,\Delta) \cup \{+\infty\} \text{ for } \epsilon > 0. \tag{12}$$

It follows from assumption 1 that

$$0 < \inf_{(k,z)\in S, a\in A_\epsilon} \tau(k,z,a) \equiv m \leq \sup_{(k,z)\in S, a\in A_\epsilon} \tau(k,z,a) \equiv M < +\infty \tag{13}$$

For $(k,z) \in S$, $a \in A_\epsilon$ define conditional probability measure Q on Borel subsets of S by

$$Q(D|(k,z),a) = P(X_{n+1} \in D|X_n = (k,z),\ a_n = a) \tag{14}$$

where X_n and a_n is the state and action at the nth decision instant. We have

$$\begin{aligned} Q(\{0,0\}|(k,z),a) &= 1 \text{ for } \epsilon \le a < \Delta \\ &= 1 - R(k,z,\Delta) \text{ for } a = +\infty \end{aligned}$$

and

$$Q(\{0,0\}|(k,z),\langle 0,a\rangle) = Q(\{0,0\}|(0,0),a) \text{ for } a \in A_\epsilon. \tag{15}$$

For any Borel set $D \subset S$, define measure γ by

$$\begin{aligned} \gamma(D) &= (1-\alpha)/M, \text{ if } \{0,0\} \in D \\ &= 0 \text{ otherwise} \end{aligned} \tag{16}$$

and let $1 > \beta > 1 - (1-\alpha)m/M$, where m and M are defined in (13).

To establish the optimality equation for restricted action space A_ϵ, it suffices to verify the following inequalities (see Kurano (1985)).

$$\begin{aligned} &Q(D|(k,z),a) \ge \tau(k,z,a)\gamma(D) \\ &\gamma(S) > (1-\beta)/\tau(k,z,a) \end{aligned} \tag{17}$$

for $(k,z) \in S$, $a \in A_\epsilon$ and for any Borel set $D \subset S$.

Obviously, if $\{0,0\} \in D$,

$$\begin{aligned} Q(D|(k,z),a) &\ge 1 - R(k,z,\Delta) \ge (1-\alpha)\tau(k,z,a)/M \\ &= \gamma(D)\tau(k,z,a) \end{aligned}$$

and if $\{0,0\} \notin D$ then $\gamma(D) = 0$ so that the first inequality in (17) holds. Next,

$$\gamma(S) = (1-\alpha)/M > (1-\beta)/m \ge (1-\beta)/\tau(k,z,a)$$

so that the second inequality in (17) is also satisfied. Kurano (1985) showed, using the idea of successive approximations, that if (17) holds then there exist a bounded function v_ϵ and a constant g_ϵ satisfying the optimality equation

$$v_\epsilon(x) = \inf_{a \in A_\epsilon} \{c(x,a) + \int v_\epsilon(x')Q(dx'|x,a) - g_\epsilon\tau(x,a)\} \tag{18}$$

where $c(x,a)$ is the expected cost incurred in the next transition interval if an action $a \in A_\epsilon$ is taken in state x. For variant 1 of our model

$$\begin{aligned} c(k,z,a) &= (K+C)(1-R(k,z,a)) + CR(k,z,a) \quad \text{for } a \in [\epsilon,\Delta) \\ c(k,z,+\infty) &= (K+C)(1-R(k,z,\Delta)) \end{aligned}$$

and

$$c(k,z,\langle 0,a\rangle) = C + c(0,0,a) \text{ for } a \in B_\epsilon. \tag{19}$$

From (18) and (19), we have

$$\begin{aligned} v_\epsilon(k,z) = \min\Big\{&C + W(0,0,+\infty,g_\epsilon,v_\epsilon),\ \inf_{a\in[\epsilon,\Delta)}\{C + V(0,0,a,g_\epsilon,v_\epsilon)\}, \\ &W(k,z,+\infty,g_\epsilon,v_\epsilon),\ \inf_{a\in[\epsilon,\Delta)}\{V(k,z,a,g_\epsilon,v_\epsilon)\}\Big\} \end{aligned} \tag{20}$$

where V and W are defined by (10) and (11), respectively. The first term on the right-hand side of (20) corresponds to an immediate replacement and a non-replacement for the next period (action $\langle 0, +\infty \rangle$), the second term corresponds to an immediate replacement and the next replacement after a time units (action $\langle 0, a \rangle$), the third term corresponds to a non-replacement and the last term to a replacement after a time units. Since $C > 0$, we have from (20)

$$v_\epsilon(0,0) = \min \left\{ W(0,0,+\infty,g_\epsilon,v_\epsilon), \inf_{a \in [\epsilon, \Delta)} \{V(0,0,a,g_\epsilon,v_\epsilon)\} \right\}. \tag{21}$$

From (20) and (21), we get

$$v_\epsilon(k,z) = \min \left\{ C + v_\epsilon(0,0),\ W(k,z,+\infty,g_\epsilon,v_\epsilon),\ \inf_{a \in [\epsilon, \Delta)} \{V(k,z,a,g_\epsilon,v_\epsilon)\} \right\}. \tag{22}$$

It can be shown, by using Lemma 1 and the approach in Kurano (1985), that $v_\epsilon(k,z)$ is nondecreasing in (k,z). The rest of the proof for variant 1 is similar to the proof of Proposition 4.1 in Kurano (1985) and can be omitted. (9) is obtained from (22) by $\lim_{\epsilon \to 0+}$, realizing that $C + v(0,0) = V(0,0,0,g,v)$.

For variant 2, since we have only 2 possible actions at each decision instant, we get

$$\begin{aligned} v^2(k,z) &= \min\{C + W(0,0,+\infty,g^2,v^2),\ W(k,z,+\infty,g^2,v^2)\}, \\ v^2(0,0) &= W(0,0,+\infty,g^2,v^2) \end{aligned} \tag{23}$$

so that (9′) holds. This completes the proof. •

In the next section, we find the form of optimal replacement policies for both variants by analyzing the optimality equations (9) and (9′).

4. FORM OF OPTIMAL REPLACEMENT POLICIES

First, we examine variant 1.

Theorem 2: *Let assumptions 1-5 be satisfied and let sequence $\{Z_n\}$ be nondecreasing with probability one. Then the optimal replacement policy f_1^* for variant 1 is a nonincreasing function of state and is given by*

$$f_1^*(k,z) = \inf\{0 \le a < \Delta : h_0(k\Delta + a)\psi(z) \ge g^1/K\} \tag{24}$$

where $f_1^*(k,z) = +\infty$ if the set in (24) is an empty set. Thus the optimal replacement rule is fully determined by g^1, which is the optimal expected average cost per unit time.

Proof: First, we shall show that if $h_0(k\Delta + a)\psi(z) < g^1/K$ for $0 \le a < \Delta$, then the optimal decision in state (k,z) is no planned replacement. By taking the derivative of $V(k,z,a,g^1,v^1)$ in (10) with respect to a, we get

$$\frac{\partial V(k,z,a,g^1,v^1)}{\partial a} = [h_0(k\Delta + a)\psi(z)K - g^1]R(k,z,a). \tag{25}$$

From (25) and from assumptions 4 and 5, we have that function V is nonincreasing for $a < a^*(k,z)$ and nondecreasing for $a \ge a^*(k,z)$, where

$$a^*(k,z) = \inf\{a \ge 0 : h_0(k\Delta + a)\psi(z) \ge g^1/K\}. \tag{26}$$

If $a^*(k,z) \ge \Delta$, then $V(k,z,\Delta,g^1,v^1) = \inf_{a \in [0,\Delta)} \{V(k,z,a,g^1,v^1)\}$ and from (10) and (11)

$$W(k,z,+\infty,g^1,v^1) - V(k,z,\Delta,g^1,v^1) = \int (v^1(k+1,y) - C - v^1(0,0))G(dy|k,z) \cdot R(k,z,\Delta) \le 0$$

so that an optimal decision is non-replacement.

Now assume that $0 \leq a^*(k,z) < \Delta$. First, we show that if $a^* = 0$ then the optimal decision is an immediate replacement.

Suppose that this is not true, *i.e.*, $a^*(k,z) = 0$ and

$$v^1(k,z) = W(k,z,+\infty,g^1,v^1) < C + v^1(0,0) = \inf_{a \in [0,\Delta)} \{V(k,z,a,g^1,v^1)\}.$$

Then from (11)

$$\begin{aligned} v^1(k+1,z) - v^1(k,z) = & \int [v^1(k+1,z) - v^1(k+1,y)] G(dy|k,z) R(k,z,\Delta) \\ & + [v^1(k+1,z) - K - C - v^1(0,0)][1 - R(k,z,\Delta)] \\ & + g^1 \tau(k,z,+\infty) \leq -K(1 - R(k,z,\Delta)) \\ & + g^1 \tau(k,z,+\infty) \leq 0. \end{aligned} \tag{27}$$

The last inequality follows from (26). So that $v^1(k+1,z) - v^1(k,z) \leq 0$ and since $v^1(k,z)$ is nondecreasing in (k,z), we must have $v^1(k+1,z) - v^1(k,z) = 0$. However, since we assumed that $v^1(k,z) < C + v^1(0,0)$ and $v^1(k+1,z) = v^1(k,z)$, we have sharp inequality in (27), so that $v^1(k+1,z) < v^1(k,z)$, which is a contradiction. Thus if $h_0(k\Delta)\psi(z) \geq g^1/K$, then $v^1(k,z) = C + v^1(0,0)$ and the optimal decision is an immediate replacement.

Now, let $0 < a^*(k,z) < \Delta$. Then $h_0((k+1)\Delta)\psi(z) \geq g^1/K$ and since $\{Z_n\}$ is nondecreasing a.s. and $\psi(\cdot)$ is nondecreasing, we have

$$W(k,z,+\infty,g^1,v^1) - V(k,z,a_1^*,g^1,v^1) = \int_{a^*(k,z)}^{\Delta} [Kh_0(k\Delta + t)\psi(z) - g^1] \cdot R(k,z,t)\, dt \geq 0$$

so that $v^1(k,z) = V(k,z,a^*,g^1,v^1)$ and an optimal decision is the replacement after $a^*(k,z)$ time units. This completes the proof. •

Now, consider variant 2.

Theorem 3: *Under the assumptions in Theorem 2, the optimal replacement policy f_2^* for variant 2 is a nonincreasing function of state and is given by*

$$\begin{aligned} f_2^*(k,z) &= +\infty \text{ if } K[1 - R(k,z,\Delta)] < g^2\tau(k,z,+\infty) \\ &= 0 \text{ if } K[1 - R(k,z,\Delta)] \geq g^2\tau(k,z,+\infty) \end{aligned}$$

where g^2 is the optimal expected average cost per unit time and $\tau(k,z,+\infty)$ is given by (8).

Proof: It follows from the optimality equation (9′) and from (11) that if $K(1 - R(k,z,\Delta)) < g^2\tau(k,z,+\infty)$ then

$$\begin{aligned} W(k,z,+\infty,g^2,v^2) - C - v^2(0,0) = & K[1 - R(k,z,\Delta)] - g^2\tau(k,z,+\infty) \\ & + \int [v^2(k+1,y) - C - v^2(0,0)] G(dy|k,z) R(k,z,\Delta) < 0 \end{aligned}$$

so that in this case $f_2^*(k,z) = +\infty$ and the optimal decision is non-replacement.

Consider the case $K(1 - R(k,z,\Delta)) \geq g^2\tau(k,z,+\infty)$ and assume that $v^2(k,z) = W(k,z,+\infty,g^2,v^2) < C + v^2(0,0)$. Then, as in (27),

$$0 \leq v^2(k+1,z) - v^2(k,z) < -K(1 - R(k,z,\Delta)) + g^2\tau(k,z,+\infty) \leq 0$$

which is a contradiction. So that if $K(1 - R(k,z,\Delta)) \geq g^2\tau(k,z,+\infty)$ then $f_2^*(k,z) = 0$ and the optimal decision is an immediate replacement. This completes the proof. •

To find optimal replacement policies for both variants, we need to compute optimal average costs g^1 and g^2. This problem will be examined in the next section.

5. COMPUTATION OF OPTIMAL POLICIES

We have from Theorem 2 and Theorem 3 that the optimal replacement rules can be found in the class of stopping times $\{T_d, d > 0\}$, where

$$T_d = \inf\{t \geq 0 : Kh(t, Z_t) \geq d\} \tag{28}$$

for variant 1 and

$$T_d = \Delta \cdot \inf\{n \geq 0 : K[1 - R(n, Z_n, \Delta)] \geq d \cdot \tau(n, Z_n, +\infty)\} \tag{29}$$

for variant 2. Then it follows from renewal theory that for both variants the expected average cost associated with replacement rule T_d is of the form

$$\phi_{T_d} = [C + KP(T_d \geq T)]/E\min(T, T_d) \tag{30}$$

where T is the time to failure of the system. We first examine the properties of function $\phi(d) \equiv \phi_{T_d}$. The approach is as in Weiss and Pliska (1982).

Theorem 4: *For both variants, function $\phi(d)$ is nonincreasing for $d \leq g^*$ and nondecreasing for $d \geq g^*$, where g^* is the optimal expected average cost. $\phi(\cdot)$ is minimized at g^*, which is the unique fixed-point of ϕ.*

Proof: We shall prove the properties of function ϕ for variant 1. The same approach can be used for variant 2. First, we show that for any $0 < x \leq y \leq z$

$$\phi(y) \leq \max\{\phi(x), \phi(z)\}. \tag{31}$$

From (30), we have

$$\begin{aligned}\phi(y) = &(\phi(x) - y)E\min(T, T_x)/E\min(T, T_y)\\ &+ \{yE\min(T, T_x) + K[P(T \leq T_y) - P(T \leq T_x)]\}/E\min(T, T_y)\end{aligned} \tag{32}$$

and

$$\begin{aligned}\phi(z) = \phi(y) + \{&K[P(T_z \geq T) - P(T_y \geq T)] + \phi(y)[E\min(T, T_y)\\ &- E\min(T, T_z)]\}/E\min(T, T_z).\end{aligned} \tag{33}$$

Let $\hat{P}(\cdot)$ and $\hat{E}(\cdot)$ be the corresponding conditional probability and expectation given a realization of the process $\{(t, Z_t),\ t \geq 0\}$ where Z_t is the value of concomitant variable at time t.

First, assume that $\phi(y) \leq y$. Then, with probability one, since T_x, T_y and T_z are stopping times with respect to process $\{(t, Z_t),\ t \geq 0\}$

$$\begin{aligned}\phi(y)(\hat{E}\min(T, T_z) - \hat{E}\min(T, T_y)) &\leq y\int_{T_y}^{T_z} \hat{P}(T > t)\,dt\\ &\leq K\int_{T_y}^{T_z} h(t, Z_t)\hat{P}(T > t)\,dt\\ &= K(\hat{P}(T \leq T_z) - \hat{P}(T \leq T_y))\end{aligned} \tag{34}$$

and from (33) and (34) we have $\phi(z) \geq \phi(y)$. The second inequality in (34) follows from the fact that for $t \geq T_y, Kh(t, Z_t) \geq y$.

Now, consider the case $\phi(y) > y$. As in (34), with probability one

$$y(\hat{E}\min(T, T_y) - \hat{E}\min(T, T_x)) \geq K(\hat{P}(T \leq T_y) - \hat{P}(T \leq T_x)) \tag{35}$$

so that we have from (32) and (35)

$$\phi(y) \leq (\phi(x) - y)E\min(T, T_x)/E\min(T, T_y) + y. \tag{36}$$

It follows from (36) that if $\phi(y) > y$ then necessarily $\phi(x) > y$ and, since $T_x \leq T_y$ a.s. for $x \leq y$, $\phi(y) \leq \phi(x)$. Thus, in both cases, $\phi(y) \leq \max\{\phi(x), \phi(z)\}$. From this and from Theorem 2, we have for any $0 < x_1 < x_2 \leq g^*$

$$\phi(x_2) \leq \max\{\phi(x_1), \phi(g^*)\} = \phi(x_1)$$

and for any $x_4 \geq x_3 \geq g^*$

$$\phi(x_3) \leq \max\{\phi(x_4), \phi(g^*)\} = \phi(x_4)$$

so that function ϕ is nonincreasing for $x \leq g^*$ and nondecreasing for $x \geq g^*$.

We know from Theorem 2 that $\phi(\cdot)$ is minimized at g^* and that g^* is a fixed point of ϕ. If p is another fixed point, then p must be greater than g^*, otherwise $\phi(p) = p < g^*$, which is a contradiction. Then from (36), setting $x = g^*, y = p$, we have

$$\phi(p) \leq (g^* - p)E\min(T, T_{g^*})/E\min(T, T_p) + p < p$$

which is again a contradiction. This completes the proof. •

To find the unique fixed-point of ϕ for both variants, we can use the following algorithm (*e.g.* Weiss and Pliska (1982), Aven and Bergman (1986)):

For any $x_1 > 0$, define

$$x_n = \phi(x_{n-1}), \text{ for } n = 2, 3, \ldots \tag{37}$$

We show that sequence $\{x_n\}$ is nonincreasing and

$$\lim_{n \to +\infty} x_n = g^* \tag{38}$$

From (36), setting $x = g^*, y = x_n$, we have for $n \geq 2$

$$x_n - x_{n+1} \geq (x_n - g^*)E\min(T, T_{g^*})/E\min(T, T_{x_n}) \geq 0 \tag{39}$$

so that $\{x_n\}$ is nonincreasing and $T_{x_n} \geq T_{x_{n+1}}$ for $n \geq 2$. From this and from (39), we can conclude that

$$x_n - x_{n+1} \geq (x_n - g^*)E\min(T, T_{g^*})/E\min(T, T_{x_2})$$

so that $\lim_{n \to +\infty} x_n = g^*$.

To apply this algorithm, it is necessary to compute $\phi(d)$ for $d > 0$, or, as it follows from (30), we need to compute $E\min\{T, T_d\}$ and $P(T_d \geq T)$.

Next, we derive a recursive computational procedure that can be used to obtain $\phi(d), d > 0$ numerically for any baseline failure distribution.

Generally, the computation of $E\min\{T, T_d\}$ and $P(T_d \geq T)$ numerically for a given $d > 0$ requires discretization. We examine variant 1. Variant 2 can be treated similarly.

Assume that $Z_n \in S = \{0, 1, 2, \ldots, m\}$ for $n \geq 0$ and let the sequence $\{t_i,\ i \in S\}$ be defined by

$$t_i = \inf\{t \geq 0 : Kh(t, i) \geq d\} \tag{40}$$

for a given $d > 0$. Let $\{k_i,\ i \in S\}$ be such integers, that

$$(k_i - 1)\Delta \leq t_i < k_i\Delta,\ i \in S. \tag{41}$$

Define for $j \geq 0$ and $r \in S$

$$W(j, r) = E[\min\{T, T_d\} - j\Delta | (j, r)] \tag{42}$$

which is the expected residual time to replacement given that the age of the system is $j\Delta$ and $Z_j = r$.
It follows from (28), (40) and (41) that

$$W(j,i) = 0 \text{ for } j \geq k_i, \tag{43}$$

$$W(k_i - 1, i) = \int_0^{t_i-(k_i-1)\Delta} R(k_i - 1, i, s)\, ds \tag{44}$$

and for $j < k_i - 1$, conditioning on the failure time yields

$$\begin{aligned} W(j,i) &= \int_{j\Delta}^{(j+1)\Delta} E(\min\{T, T_d\}|(j,i), T = s)\, F(d(s - j\Delta)|(j,i)) \\ &+ \sum_{r=i}^{m} E(\min\{T, T_d\}|(j,i),\ T > (j+1)\Delta, Z_{j+1} = r) \\ &\cdot P[T > (j+1)\Delta, Z_{j+1} = r|(j,i)] - j\Delta \\ &= \int_{j\Delta}^{(j+1)\Delta} (s - j\Delta)\, F(d(s - j\Delta)|(j,i)) + \Delta R(j,i,\Delta) + \\ &+ \sum_{r=i}^{m} [E(\min\{T, T_d\}|(j+1, r)) - (j+1)\Delta] \cdot P_{i,r}(j) R(j,i,\Delta) \\ &= \int_0^{\Delta} R(j,i,s) ds + \sum_{r=i}^{m} W(j+1, r)\, P_{i,r}(j) R(j,i,\Delta) \end{aligned} \tag{45}$$

where

$$P_{i,r}(j) = P(Z_{j+1} = r|T > (j+1)\Delta, Z_j = i), \tag{46}$$

$F(s|(j,i)) = 1 - R(j,i,s)$.
The backward recursion (43)-(45) can be used to obtain

$$W(0,0) = E(\min\{T, T_d\}).$$

A similar procedure can be derived to obtain the probability $P(T_d \geq T)$. Denote for $i \in S$

$$Q(j,i) = P(T_d \geq T|(j,i)) \tag{47}$$

and let t_i and k_i be given by (40) and (41), respectively. Then

$$Q(j,i) = 0 \text{ for } j \geq k_i \tag{48}$$

$$Q(k_i - 1, i) = 1 - R(k_i - 1, i, t_i - (k_i - 1)\Delta) \tag{49}$$

and for $j < k_i - 1$

$$Q(j,i) = 1 - R(j, i\Delta) + \sum_{r=i}^{m} Q(j+1, r) P_{i,r}(j) R(j,i,\Delta) \tag{50}$$

where $P_{i,r}(j)$ is given by (46). The probability $P(T_d \geq T) = Q(0,0)$. Since the hazard rate $h(t,z)$ in (1) is assumed to be nondecreasing in both t and z, it follows from (40) that the sequence $\{t_i,\ i \in S\}$ is nonincreasing,

$$t_0 \geq t_1 \geq \cdots \geq t_m \tag{51}$$

To illustrate the recursive procedures (43)-(45) and (48)-(50), we consider the following example.

Example: Assume that the baseline distribution is a Weibull distribution with hazard function

$$h_0(t) = \frac{\beta t^{\beta-1}}{\alpha^\beta}, \quad t \geq 0$$

Table 1. An Illustration of the Computational Procedure for Variant 1

x_i	t_0	t_1	k_0	k_1	$W(0,0)$	$Q(0,0)$	$\phi(x_i)$
5	1.250	0.760	2	1	0.7750	0.695	8.245
8.245	2.060	1.250	3	2	0.8350	0.905	8.160
8.160	2.040	1.240	3	2	0.8346	0.902	8.150
8.150	2.038	1.236	3	2	0.8342	0.901	8.150

where $\alpha = 1$, $\beta = 2$ and let $\psi(z) = e^{0.5z}$, $K = 2$, $C = 5$, $\Delta = 1$. We assume that $\{Z_n, n \geq 0\}$ is a homogeneous Markov chain with two states $\{0, 1\}$ and with the transition probability matrix

$$P = \begin{pmatrix} 0.4 & 0.6 \\ 0 & 1 \end{pmatrix}$$

We start *e.g.* with value $x_1 = 5$ in (37) and find t_0 and t_1 from (40). We get $t_0 = 1.25$, $t_1 = 0.758$ and from (41), $k_0 = 2$, $k_1 = 1$. Further results are in Table 1.

From Table 1, the optimal expected average cost $g^* = 8.15$ and the optimal replacement time is given by

$$T^* = \min\{T, \inf\{t \geq 0 : 4 \cdot t \cdot e^{0.5Z_t} \geq 8.15\}$$

6. CONCLUSIONS AND SUMMARY

We have studied an optimal replacement problem for a deteriorating system subject to random failure. The proportional hazards model has been used to describe the failure rate of the system which is a function of age but can also depend on values of a diagnostic stochastic process. It has been assumed that the process can be observed only at discrete points of time. In engineering applications this additional information can usually be obtained through inspections. The aim of the paper has been to show how this information can be utilized to improve decision-making in preventive replacement. Two variants of the model have been considered. In variant 1, the system can be preventively replaced at any time and in variant 2, the system is available for preventive replacement only at discrete points of time. The form of the optimal replacement policy has been established for both variants directly by analyzing the optimality equations.

A computational procedure based on a backward recursion has been developed which can be applied to obtain the optimal average cost and the optimal policy for any baseline failure distribution.

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