TAREA COMPLEMENTOS DE EDP, 2010/1

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(1) Consider the energy functional, defined for functions $u \in H^1(\mathbb{R})$,

$$J_\varepsilon(u) = \int_{\mathbb{R}} \left[\varepsilon^2 \frac{|u'|^2}{2} + \frac{u^2}{2} - \frac{u^4}{4} \right] a(x)\, dx$$

where $a(x)$ is a positive smooth function, with bounded derivatives, and with $\inf_{\mathbb{R}} a > 0$.

(i) (1 point). Prove that a critical point u of J_ε in $H^1(\mathbb{R})$ is a classical solution of the problem

$$\varepsilon^2 [a(x)u']' + a(x)\,[u^3 - u] \;=\; 0, \quad u(\pm\infty) = 0. \tag{0.1}$$

Let

$$w(t) \;=\; \frac{\sqrt{2}}{\cosh t}$$

which we observe it solves

$$w'' - w + w^3 = 0, \quad w > 0, \quad w(\pm\infty) = 0.$$

(ii) (5 points) Prove the following result (essentially due to Floer and Weinstein, 1986):

Let $\bar{x} \in \mathbb{R}$ be a point such that $a'(\bar{x}) = 0$ and $a''(\bar{x}) \neq 0$. Then there exists a "spike solution of Problem (0.1) concentrating near $\bar{x}$", namely a positive solution u_ε such that

$$u_\varepsilon(x) = w\left(\frac{x - \bar{x}}{\varepsilon}\right) + \theta_\varepsilon(x)$$

where $\theta_\varepsilon \to 0$ uniformly on $\mathbb{R}$ as $\varepsilon \to 0$.

Hint: In Equation (0.1) write the equation in terms of $v_\varepsilon(t) := u_\varepsilon(\bar{x} + \varepsilon(t + h))$ for a given $h \in \mathbb{R}$. Then look for a solution of the form

$$v_\varepsilon(t) = w(t) + \phi(t)$$

for a small function ϕ. You will need to solve a linear problem of the form

$$\psi'' + (3w^2(t) - 1)\psi = g(t) - cw'(t), \quad c := \frac{\int_{-\infty}^{\infty} g(t)\, w'(t)\, dt}{\int_{-\infty}^{\infty} w'(t)^2\, dt}.$$

Show that if $g \in L^\infty(\mathbb{R})$ this problem has a unique solution $\psi = T(g) \in L^\infty(\mathbb{R})$ with $\psi'(0) = 0$. Moreover, we have

$$\|T(g)\|_{L^\infty(\mathbb{R})} \;\leq\; C\, \|g\|_{L^\infty(\mathbb{R})}.$$

(2)

a) Consider the equation

$$-\Delta u = \lambda e^u \quad \text{in } B_1$$
$$u = 0 \quad \text{on } \partial B_1$$

where $\lambda > 0$ and B_1 is the unit ball in $\mathbb{R}^N$, $N \geq 3$.

Show that there exists a radial singular solution u_s associated to the parameter $\lambda_s = 2(N-2)$. We assume now that $3 \leq N \leq 9$. Prove that for $\lambda = \lambda_s$ there are infinitely many solutions, and that the number of solutions tends to infinity as $\lambda \to \lambda_s$. Show also that $\lambda_s < \lambda^*$ and u^* is bounded, where λ^* is the extremal parameter (for existence of a bounded solution), and $u^* = \lim_{\lambda \to \lambda^*} u_\lambda$, where u_λ is the minimal solution.

Prove also that if $\lambda_k \to \lambda_s$ and u_k is a solution with parameter λ_k such that $\sup_{B_1} u_k \to \infty$ as $k \to \infty$, then the Morse index of u_k tends to infinity as $k \to \infty$.

b) Prove that if $3 \leq N \leq 9$ there are no stable solutions of

$$-\Delta u = e^u \quad \text{in } \mathbb{R}^N$$

that are bounded above.

For a), let $v(t) = u(r)$, $r = e^t$, and find the equation satisfied by v. Then let

$$v_1(t) = \frac{\lambda}{2(N-2)} e^{v(t)+2t}, \quad v_2(t) = v'(t).$$

Find the system of ODE satisfied by v_1, v_2. Show that there are 2 stationay points (assume always $N \geq 3$), the origin and $P \neq 0$.

Show that the smooth radial solution to the problem

$$-\Delta U = 2(N-2)e^U \quad \text{in } \mathbb{R}^N$$

gives rise to a solution of the system for v_1, v_2, that is a heteroclinic connection from 0 to P. For this it is useful to consider another change of variable: $w(t) = v(t) + 2t$. Then find the ODE satisfied by w and show that along trajectories the following energy decreases

$$E(t) = \frac{1}{2} w'(t)^2 + 2(N-2)(e^{w(t)} - 1).$$

Find the linearization around P and show that if $3 \leq N \leq 9$ then the eigenvalues of this linearization have nonzero imaginary part. Deduce the statements about multiplicity.

To establish that the Morse of u_k tends to infinity as $k \to \infty$ (in the unit ball and $3 \leq N \leq 9$), show that the number of intersections of u_k with the singular solution tends to infinity as $k \to \infty$. Use the difference $u_s - u_k$ restricted to appropriate intervals as test-functions for which the quadratic form associated to stability is negative.

For b), use $e^{\alpha u} \eta^2$ as a test function in the equation where $\eta \in C_c^\infty(\mathbb{R}^N)$ and $\alpha > 0$. Then use a similar test function in the stability condition of u. For $\beta > 0$ in an appropriate range you will be able to deduce the behavior of $\int_{B_R(0)} e^{\beta u}$ as $R \to \infty$.