Solución del problema 3

Teníamos que

$$\begin{aligned} y_p &= -\sin x \int \frac{\cos x \tan x}{-1} dx + \cos x \int \frac{\sin x \tan x}{-1} dx \\ &= -\sin x \int -\sin x dx + \cos x \int \frac{-\sin^2 x}{\cos x} dx \end{aligned}$$

Ahora,

$$\int \frac{\sin^2 x}{\cos x} dx = \int \frac{1}{\cos x} dx - \int \cos x dx$$

y

$$\begin{aligned} \int \frac{1}{\cos x} dx &= \int \frac{\cos x dx}{1-\sin^2 x} \\ &= \int \frac{1}{1-u^2} du \\ &= \frac{1}{2} \ln \left| \frac{1-u}{1+u} \right| \\ &= \frac{1}{2} \ln \left| \frac{1-\sin x}{1+\sin x} \right| \\ &= \frac{1}{2} \ln \left| \frac{(1-\sin x)^2}{1-\sin^2 x} \right| \\ &= \ln \left| \frac{1-\sin x}{\cos x} \right| \\ &= \ln |\sec x + \tan x| \\ &= \ln \frac{1+\sin x}{\cos x} \end{aligned}$$

de modo tal que

$$\begin{aligned} y_p &= -\sin x \cos x - \cos x \left(\ln \frac{1+\sin x}{\cos x} - \sin x \right) \\ &= -\cos x \ln \left(\frac{1+\sin x}{\cos x} \right) \\ \Rightarrow y_p' &= \sin x \ln \left(\frac{1+\sin x}{\cos x} \right) - 1 \end{aligned}$$

de modo que

$$y(x) = A\cos x + B\sin x + y_p(x).$$

Imponiendo $y(0) = 0$, y como $y_p(0) = 0$, sigue que $A = 0$. Imponiendo $y'(0) = 1, B = 2$, y la solución del problema es

$$y(x) = 2\sin x - \cos x \ln \left(\frac{1+\sin x}{\cos x} \right)$$