Pauta P2 Control 3 MA 2601, 2010/1

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Duración 3 hrs.

2. *a*) Resuelva las siguientes ecuaciones usando transformada de Laplace:

   1) (1pt) $y' = \sin(t) + \int_0^t y(t-\tau)\cos(\tau)d\tau$, $y(0) = 0$.
   2) (1pt) $y'' + 4y = e^{-2t}$, $y(0) = 0$, $y'(0) = 1$.
   3) (1pt) $y'' + 4y' + 4y = f(t)$, $y(0) = y'(0) = 0$, donde $f(t) = t$ si $0 \leq t \leq 1$ y $f(t) = 0$ para $t > 1$.

   b) (3pt) Considere la ecuación

$$x'' + x = \sum_{n=0}^{\infty} (-1)^n \delta_{n\pi}, \quad x(0) = x'(0) = 0.$$

Determine la solución de esta ecuación y esboce el gráfico de la solución.
Indicación: Intercambie la serie con la transformada y antitransformada.

Solución:

2. *a*) 1) Aplicamos transformada de Laplace a la ecuación,

$$\begin{aligned}
\mathcal{L}(y') &= \mathcal{L}(\sin t) + \mathcal{L}\left(\int_0^t y(t-\tau)\cos(\tau)d\tau\right)(s) & \Leftrightarrow \\
s\mathcal{L}(y) - y(0) &= \frac{1}{1+s^2} + \mathcal{L}(y * \cos(t))(s) & \Leftrightarrow \\
s\mathcal{L}(y) &= \frac{1}{1+s^2} + \mathcal{L}(y) \cdot \mathcal{L}(\cos(t))(s) & \Leftrightarrow \\
s\mathcal{L}(y) &= \frac{1}{1+s^2} + \mathcal{L}(y) \cdot \frac{s}{s^2+1} & \Leftrightarrow \\
\left(s - \frac{s}{s^2+1}\right)\mathcal{L}(y) &= \frac{1}{1+s^2} & \Leftrightarrow \\
\mathcal{L}(y) &= \frac{1}{s^3} & \Leftrightarrow \\
y(t) &= \frac{t^2}{2}.
\end{aligned}$$

2) Aplicando transformada de Laplace a la ecuación,

$$\begin{aligned}
\mathcal{L}(y'') + 4\mathcal{L}(y) &= \mathcal{L}(e^{-st}) & \Leftrightarrow \\
s^2\mathcal{L}(y)(s) - sy(0) - y'(0) + 4\mathcal{L}(y)(s) &= \mathcal{L}(e^{-st})(s) & \Leftrightarrow \\
s^2\mathcal{L}(y)(s) - 1 + 4\mathcal{L}(y)(s) &= \mathcal{L}(e^{-st})(s) & \Leftrightarrow \\
(s^2+1)\mathcal{L}(y)(s) &= 1 + \mathcal{L}(e^{-st})(s) & \Leftrightarrow \\
\mathcal{L}(y)(s) &= \frac{1}{s^2+1} + \mathcal{L}(e^{-st})(s)\frac{1}{s^2+1} & \Leftrightarrow \\
\mathcal{L}(y)(s) &= \mathcal{L}(\sin(t))(s) + \mathcal{L}(e^{-st})(s)\mathcal{L}(\sin(t))(s) & \Leftrightarrow \\
\mathcal{L}(y)(s) &= \mathcal{L}(\sin(t))(s) + \mathcal{L}\left(e^{-st} * \sin(t)\right)(s) & \Leftrightarrow \\
y(t) &= \sin(t) + e^{-st} * \sin(t)
\end{aligned}$$

3) Aplicando transformada de Laplace a la ecuación,

$$\begin{aligned}
\mathcal{L}(y'') + 4\mathcal{L}(y') + 4\mathcal{L}(y) &= \mathcal{L}(f(t)) && \Leftrightarrow \\
s^2\mathcal{L}(y) - sy(0) - y'(0) + 4s\mathcal{L}(y) - 4y(0) + 4\mathcal{L}(y) &= \mathcal{L}(f(t)) && \Leftrightarrow \\
s^2\mathcal{L}(y)(s) + 4s\mathcal{L}(y)(s) + 4\mathcal{L}(y)(s) &= \mathcal{L}(f(t))(s) && \Leftrightarrow \\
(s^2+4s+4)\mathcal{L}(y)(s) &= \mathcal{L}(f(t))(s) && \Leftrightarrow \\
\mathcal{L}(y)(s) &= \mathcal{L}(f(t))(s)\frac{1}{s^2+4s+4} && \Leftrightarrow \\
\mathcal{L}(y)(s) &= \mathcal{L}(f(t))(s)\frac{1}{(s+2)^2} && \Leftrightarrow \\
\mathcal{L}(y)(s) &= \mathcal{L}(f(t))(s)\mathcal{L}(te^{-2t})(s) && \Leftrightarrow \\
\mathcal{L}(y)(s) &= \mathcal{L}\left(f(t) * te^{-2t}\right)(s) && \Leftrightarrow \\
y(t) &= f(t) * te^{-2t}.
\end{aligned}$$

Donde hemos usado que $\frac{1}{(s+2)^2} = \mathcal{L}(te^{-2t})(s)$, en efecto:

$$\begin{aligned}
\frac{1}{(s+2)^2} &= -\frac{d}{ds}\frac{1}{(s+2)} && \Leftrightarrow \\
&= -\frac{d}{ds}\mathcal{L}(1)(s+2) && \Leftrightarrow \\
&= -\frac{d}{ds}\mathcal{L}\left(e^{-2t}\right)(s) && \Leftrightarrow \\
&= \mathcal{L}\left(te^{-2t}\right)(s)
\end{aligned}$$

b) Aplicamos transformada de Laplace,

$$
\begin{aligned}
\mathcal{L}(x'') + \mathcal{L}(y) &= \mathcal{L}\left(\sum_{n=0}^{\infty}(-1)^n \delta_{n\pi}\right) && \Leftrightarrow \\
s^2\mathcal{L}(x) - sx(0) - x'(0) + \mathcal{L}(x) &= \mathcal{L}\left(\sum_{n=0}^{\infty}(-1)^n \delta_{n\pi}\right) && \Leftrightarrow \\
s^2\mathcal{L}(y)(s) + \mathcal{L}(y)(s) &= \mathcal{L}\left(\sum_{n=0}^{\infty}(-1)^n \delta_{n\pi}\right) && \Leftrightarrow \\
(s^2+1)\mathcal{L}(y)(s) &= \mathcal{L}\left(\sum_{n=0}^{\infty}(-1)^n \delta_{n\pi}\right) && \Leftrightarrow \\
\mathcal{L}(y)(s) &= \mathcal{L}\left(\sum_{n=0}^{\infty}(-1)^n \delta_{n\pi}\right)\frac{1}{s^2+1} && \Leftrightarrow \\
\mathcal{L}(y)(s) &= \mathcal{L}\left(\sum_{n=0}^{\infty}(-1)^n \delta_{n\pi}\right)\mathcal{L}(\sin(t)) && \Leftrightarrow \\
\mathcal{L}(y)(s) &= \mathcal{L}\left(\left(\sum_{n=0}^{\infty}(-1)^n \delta_{n\pi}\right) * \sin(t)\right) && \Leftrightarrow \\
\mathcal{L}(y)(s) &= \mathcal{L}\left(\sum_{n=0}^{\infty}(-1)^n \delta_{n\pi} * \sin(t)\right) && \Leftrightarrow \\
y(t) &= \sum_{n=0}^{\infty}(-1)^n \delta_{n\pi} * \sin(t) && \Leftrightarrow \\
y(t) &= \sum_{n=0}^{\infty} \sin(t) H(t - n\pi) && \Leftrightarrow
\end{aligned}
$$

En la figura 1 se encuentra el gráfico de $y(t)$.

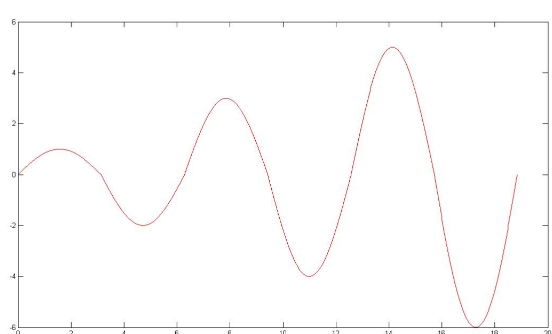

Figura 1: Gráfico de $y(t)$.