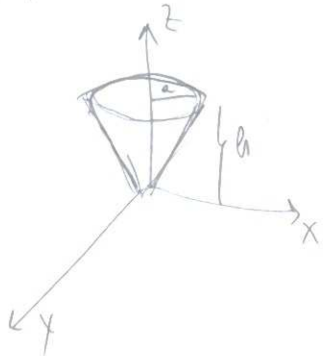
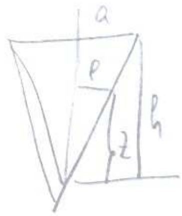
P1 manto de un cono. $\longrightarrow$ Determinar A.



1º <u>Parametrizar</u>



$$\frac{\rho}{z} = \frac{a}{h} \Rightarrow \boxed{z = \frac{h}{a}\rho \quad 0 \le \rho \le a}$$

$$\vec{r} = \rho\hat{\rho} + z\hat{k} = \vec{r}(\rho, \theta, z)$$

$$\Rightarrow \vec{r}(\rho,\theta) = \vec{r}\left(\rho, \theta, \frac{h}{a}\rho\right) = \rho\hat{\rho} + \frac{h}{a}\rho\hat{k} \qquad \begin{array}{l}\rho \in [0,a] \\ \theta \in [0, 2\pi)\end{array}$$

Luego, queremos

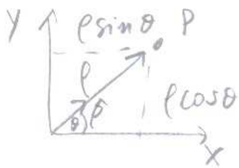
$$\iint_S dA = \iint_D \left\| \frac{\partial r}{\partial \rho} \times \frac{\partial r}{\partial \theta} \right\| d\rho\, d\theta$$

$$\frac{\partial r}{\partial \rho} = \frac{\partial}{\partial \rho}\left(\rho\hat{\rho}(\theta) + \frac{h}{a}\rho\hat{k}\right) = \hat{\rho}(\theta) + \frac{h}{a}\hat{k}$$

$$\frac{\partial r}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\rho\hat{\rho}(\theta) + \underbrace{\frac{h}{a}\rho\hat{k}}_{\text{indep. de } \theta}\right) = \frac{\partial}{\partial \theta}\left(\rho\hat{\rho}(\theta)\right)$$

pero $\hat{\rho} = (\cos\theta, \sin\theta, 0)$

$\frac{\partial \hat{\rho}}{\partial \theta} = (-\sin\theta, \cos\theta, 0) = \hat{\theta}$



$\Rightarrow P = (x, y) = (\rho\cos\theta, \rho\sin\theta) = \rho\hat{\rho}$.

$$\Rightarrow \frac{\partial}{\partial \theta}\left(\rho\hat{\rho}(\theta)\right) = \rho\frac{\partial}{\partial\theta}\hat{\rho} = \rho\hat{\theta} = \frac{\partial \vec{r}}{\partial \theta}.$$

$\hat{\rho} \to \hat{\theta} \to \hat{k}$

$$\Rightarrow \left\| \frac{\partial \vec{r}}{\partial \rho} \times \frac{\partial \vec{r}}{\partial \theta} \right\| = \left\| \left(\hat{\rho} + \frac{h}{a}\hat{k}\right) \times \left(\rho\hat{\theta}\right) \right\| =$$

$$\left\| \rho\hat{k} - \rho\frac{h}{a}\hat{\rho} \right\| = \rho\sqrt{1 + \left(\frac{h}{a}\right)^2}$$

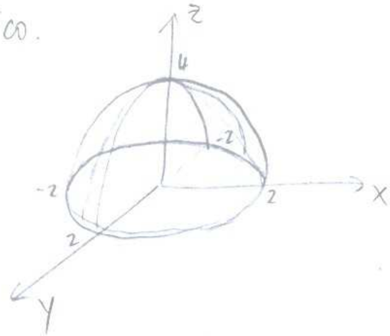
$$\Rightarrow A = \int_0^{2\pi}\int_0^a \rho\sqrt{1 + \left(\frac{h}{a}\right)^2}\, d\rho\, d\theta = \int_0^a 2\pi\rho\sqrt{1 + \left(\frac{h}{a}\right)^2}\, d\rho = 2\pi\sqrt{1 + \left(\frac{h}{a}\right)^2}\int_0^a \rho\, d\rho$$

$$= 2\pi\sqrt{1 + \left(\frac{h}{a}\right)^2} \cdot \frac{a^2}{2} = \pi a\sqrt{a^2 + h^2}$$

//

P2 | Sea la sup. def. por $z = 4 - x^2 - y^2$, $z \geq 0$

i) Gráfico.



Si $z=0$? $\longrightarrow$ circunf. $(r=2)$

$x=0 \longrightarrow$ parab. en yz

$y=0 \longrightarrow$ parab en xz

$\Rightarrow$ Paraboloide!

$$z_G = \frac{1}{M}\int_S z\,dm = \frac{1}{M}\int_S z\,\underbrace{\sigma\,dA}_{dm}$$

b) Det. CG

$$\left.\begin{aligned} x_G &= \frac{1}{M}\iint x\cdot\sigma\,dA = 0 \\ y_G &= \frac{1}{M}\iint x\cdot\sigma\,dA = 0 \end{aligned}\right\} \text{ simetría} \qquad z_G = \frac{1}{M}\iint_S z\cdot\sigma\,dA \neq 0$$

σ = densidad superficial (i.e. $[M]/A = [M]/[L^2]$)

1° Parametrizar $\vec{r}(\rho,\theta) = (\rho\cos\theta, \rho\sin\theta, 4-\rho^2)$ $\quad \theta \in [0, 2\pi)$, $\rho \in [0, 2]$

$$\left.\begin{aligned} \frac{\partial \vec{r}}{\partial \rho} &= (\cos\theta, \sin\theta, -2\rho) \\ \frac{\partial \vec{r}}{\partial \theta} &= (-\rho\sin\theta, \rho\cos\theta, 0) \end{aligned}\right\} \frac{\partial \vec{r}}{\partial \rho}\times\frac{\partial \vec{r}}{\partial \theta} = (2\rho^2\cos\theta, 2\rho^2\sin\theta, \rho)$$

$$\| \,\%\, \| = \sqrt{4\rho^4\cos^2\theta + 4\rho^4\sin^2\theta + \rho^2} = \rho\sqrt{1+4\rho^2}$$

$$\Rightarrow dA = \sqrt{1+4(x^2+y^2)}\,dy\,dx = \sqrt{1+4\rho^2}\,\rho\,d\rho\,d\theta$$

$\Rightarrow$ ¿Y σ? Calculemos la masa $\quad M = \iint_S \sigma\,dA$

$$M = \int_0^{2\pi}\int_0^2 \sigma\sqrt{1+4\rho^2}\,\rho\,d\rho\,d\theta = 2\pi\sigma\int_0^2 \rho\sqrt{1+4\rho^2}\,d\rho \qquad \text{Pero: } \left(\frac{2}{3}(1+4\rho^2)^{3/2}\cdot\frac{1}{8}\right)' = \rho\sqrt{1+4\rho^2}$$

$$= 2\pi\sigma\left[\frac{2}{3}(1+4\rho^2)^{3/2}\cdot\frac{1}{8}\right]_0^2 = 2\pi\sigma\left[\frac{\cancel{2}}{3}\cdot 17^{3/2}\cdot\frac{1}{\cancel{8}4} - \frac{1}{12}\right]$$

$$M = \frac{\pi\sigma}{6}\left[17^{3/2} - 1\right]$$

$$\Rightarrow z_G = \frac{1}{\frac{\pi\cancel{\sigma}}{6}(17^{3/2}-1)}\int_0^{2\pi}\int_0^2 (4-\rho^2)\cdot\cancel{\sigma}\sqrt{1+4\rho^2}\,\rho\,d\rho\,d\theta = \frac{6\cdot 2\cancel{\pi}}{\cancel{\pi}(17^{3/2}-1)}\underbrace{\int_0^2 (4-\rho^2)\rho\sqrt{1+4\rho^2}\,d\rho}_{I(\rho)} = \frac{12}{(17^{3/2}-1)}\cdot I(\rho)$$

$z_G = \frac{1}{M}\iint \rho\,d$

$z_G = \frac{1}{M}\int z\,(dm)$ $\sigma\,dA$

P3) a) Calcular el $\acute{A}$ de la parte del cono $x^2+y^2=z^2$, $z \geq 0$ que está dentro de la esfera $x^2+y^2+z^2=2Rz$ con R cte >0

b) Calcular el $\acute{A}$ de la parte de la esf. que está dentro del cono.

Sol. 1^{ero} reescribamos: $x^2+y^2+z^2=2Rz \Leftrightarrow x^2+y^2+\underbrace{(z^2-2Rz+R^2)}=R^2$

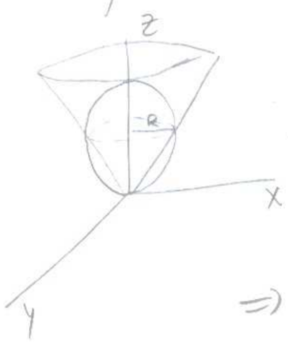
$x^2+y^2+(z-R)^2=R^2$

2^{do} Determinemos la $\cap$:

i.e. $x^2+y^2=z^2 \cap x^2+y^2+(z-R)^2=R^2$

$\Rightarrow z^2+z^2-2Rz+\cancel{R^2}=\cancel{R^2} \Rightarrow \cancel{2}z\cancel{^2}=\cancel{2}R\cancel{z} \Rightarrow \boxed{z=R}$

es decir, a partir de $z=R$ y bajando hacia 0, el cono está dentro de la esfera.



Entonces: $\vec{r}=(\rho\cos\theta, \rho\sin\theta, \rho)$ $\rho=z$

con $0 \leq \theta \leq 2\pi$

$0 \leq \rho \leq R$.

$\Rightarrow \frac{\partial \vec{r}}{\partial \rho}=(\cos\theta, \sin\theta, 1)$ $\frac{\partial \vec{r}}{\partial \theta}=(-\rho\sin\theta, \rho\cos\theta, 0)$

$$\Rightarrow \frac{\partial \vec{r}}{\partial \rho} \times \frac{\partial \vec{r}}{\partial \theta} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 1 \\ -\rho\sin\theta & \rho\cos\theta & 0 \end{vmatrix} = -\rho\cos\theta\hat{i}+\rho\sin\theta\hat{j}+\rho(\cos^2\theta+\sin^2\theta)\hat{k}$$

$$= (-\rho\cos\theta, \rho\sin\theta, \rho)$$

$$\Rightarrow dA = \left\| \frac{\partial \vec{r}}{\partial \rho} \times \frac{\partial \vec{r}}{\partial \theta} \right\| d\rho\, d\theta = \sqrt{\rho^2\cos\theta+\rho^2\sin\theta+\rho^2} = \rho\sqrt{2}\, d\rho\, d\theta$$

$$\Rightarrow A = \sqrt{2}\int_0^{2\pi}\int_0^R \rho\, d\rho\, d\theta = \cancel{2}\pi\sqrt{2}\cdot\frac{R^2}{\cancel{2}} = \pi\sqrt{2}R^2$$

b) Como la $\cap$ se produce para $z=R$, es directo que la mitad de la esf queda dentro del cono

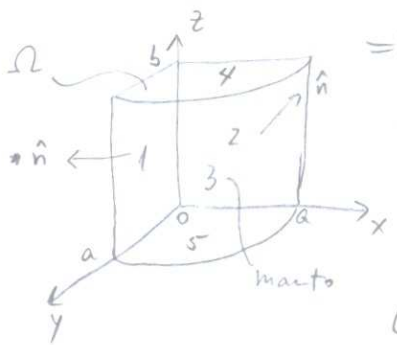
$$\Rightarrow A = \frac{1}{2}A_{esf} = \frac{1}{\cancel{2}}\cancel{4}\pi R^2 = 2\pi R^2$$

P4) Sea el campo $\vec{F}$ dado por: $\vec{F} = (yz, xz, xy)$

y la reg. $\Omega = \{x \geq 0, y \geq 0, z \in [0,b], x^2+y^2 \leq a^2, a \wedge b > 0\}$.

Calcule el flujo del campo $\vec{F}$ a través de Ω. (Considerando la normal exterior)

Sol 1ero Grafiquemos $\Omega. = \{x \geq 0, y \geq 0, \underbrace{z \in [0,b]}_{z \text{ variable.}}, \underbrace{x^2+y^2 \leq a^2}_{\text{circ. de radio} \leq a}, a, b > 0\}$.



$\Rightarrow \Omega$ es $\frac{1}{4}$ cilindro de altura b y radio basal a

Separemos $\partial\Omega$ por caras i.e. $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2 \cup \partial\Omega_3 \cup \partial\Omega_4 \cup \partial\Omega_5$

Luego: $\iint_{\partial\Omega} \vec{F}\cdot\hat{n}\,dA = \sum_{i=1}^{5} \iint_{\partial\Omega_i} \vec{F}\cdot d\vec{S}$. $\quad f_i = \iint_{\partial\Omega_i} \vec{F}\cdot d\vec{S}$.

Calculemos los flujos

$f_1 = \iint_{\partial\Omega_1} \vec{F}\cdot d\vec{S} = \iint_{\partial\Omega_1} \vec{F}\cdot\hat{n}\,dA \quad \hat{n} = -\hat{\imath} = (-1,0,0) \quad dA = dy\,dz$. (x es fijo!)

$\vec{F} = y\hat{\jmath} + z\hat{k} \quad y \in [0,a] \quad z \in [0,b]$

$$\Rightarrow f_1 = \int_0^b\int_0^a \vec{F}\cdot\hat{n}\,dA = \int_0^b\int_0^a (yz, xz, xy)\cdot(-1,0,0)\,dy\,dz$$

$$= \int_0^b\int_0^a -yz\,dy\,dz = \boxed{-\frac{a^2b^2}{4} = f_1}$$

$f_2 = \iint_{\partial\Omega_2} \vec{F}\cdot\hat{n}\,dA \quad \hat{n} = (0,-1,0) \quad dA = dx\,dz. \Rightarrow f_2 = \int_0^a\int_0^b (yz, xz, xy)(0,-1,0)\,dx\,dz$

$f_2 = \int_0^b\int_0^a -xz\,dx\,dz = -\frac{a^2b^2}{4}$ (tb. directo por simetría).

$f_3 = \iint_{\partial\Omega_3} \vec{F}\cdot\hat{n}\,dA$ en este caso: $\vec{r} = (a\cos\theta, a\sin\theta, z) \underset{\rho = a \text{ fijo}}{\Rightarrow} \begin{cases} \frac{\partial r}{\partial\theta} = (-a\sin\theta, a\cos\theta, 0) \\ \frac{\partial r}{\partial z} = (0,0,1) \end{cases}$

$\Rightarrow \frac{\partial r}{\partial\theta}\times\frac{\partial r}{\partial z} = (a\cos\theta, a\sin\theta, 0) \wedge \|\%\| = \sqrt{a^2} = a \Rightarrow \hat{n} = \frac{(a\cos\theta, a\sin\theta, 0)}{a} = (\cos\theta, \sin$

$= \hat{\rho}$ (obvio)

$dA = a\,d\theta\,dz \Rightarrow f_3 = \int_0^{\pi/2}\int_0^b (az\sin\theta, az\cos\theta, a^2\sin\theta\cos\theta)\cdot(\cos\theta, \sin\theta, 0)\,a\,dz\,d\theta$. $\quad \theta \in [0, \pi/2]$, $z \in [0,b]$

$$= \int_0^{\pi/2}\int_0^b (a^2z\sin\theta\cos\theta + a^2z\sin\theta\cos\theta)\,dz\,d\theta = \int_0^{\pi/2}\int_0^b a^2z\sin 2\theta\,d\theta\,dz = \frac{a^2b^2}{2}\int_0^{\pi/2}\sin 2\theta\,d\theta$$

pero: $\int \sin 2\theta = -\frac{\cos 2\theta}{2} \Rightarrow f_3 = \frac{a^2b^2}{2}\left[-\frac{\cos 2\theta}{2}\right]\Big|_0^{\pi/2} = \frac{a^2b^2}{2}$. $\quad f_4 = -f_5$ (simetría)

Luego: $f_{TOT} = \iint \vec{F}\cdot d\vec{S} = \sum f_i = 0$.