A LINE-INTEGRAL METHOD OF COMPUTING THE GRAVIMETRIC EFFECTS OF TWO-DIMENSIONAL MASSES*

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ABSTRACT

Many computing schemes have been devised for determining the gravity anomalies produced by two-dimensional masses. Most of these are based upon the evaluation of an areal integral and require specially constructed templates or tables. In the present paper it is shown that the gravity anomaly Δg at the origin of coordinates, produced by a two-dimensional mass of constant density contrast $\Delta\rho$, may be obtained quite simply by means of either of the line integrals

$$\Delta g = 2k\Delta\rho \oint \theta dz = -2k\Delta\rho \oint z d\theta,$$

where z is the vertical coordinate, and θ the polar coordinate expressed in radians of a point on the periphery of the mass in a plane normal to its axis and passing through the origin

The line integrals are evaluated around the periphery of the mass and are of opposite sign if taken in the same direction of traverse, or are of the same sign if taken in opposite directions.

For use of these integrals no special equipment is required other than a simple template consisting of radial lines, θ=const., and horizontal lines, z=const., which can be constructed in a few minutes with protractor and scale. This can be constructed either for 1:1 or for an exaggerated vertical-to-horizontal scale.

A mass distribution is said to be two-dimensional when its density varies as the same function of position on each of a family of parallel planes. If one of these planes is chosen as the xz-plane in a system of cartesian coordinates, then the density would be an arbitrary function of x and z, but would be constant along any line parallel to the y-axis. The computation of the gravitational effects of such distributions is much simpler than for three-dimensional distributions, since in the former case the integration is limited to a plane, whereas in the latter case it must be performed over a volume of space.

Computations of the effects of two-dimensional masses are of considerable interest in gravimetric prospecting because many of the most common of geologic structures—folding parallel to a given horizontal axis, parallel faulting, ridge-and-valley topography resulting from the erosion of folded structures, etc.,—can be approximated by such distributions.

Many computing schemes for the effects produced by two-dimensional masses have been described in the literature. Mostly these require the use of special templates, mechanical devices, or tables of functions. Moreover, they usually are not applicable to masses extending to infinity or lying too near the horizon of the station, consequently, the existing methods are often not well suited to computing problems encountered in practice where the masses often do extend to infinity, and, in such problems as terrain connections, also lie near the horizon. A

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need has therefore been felt for a simpler and more versatile method of making such calculations, which has satisfactorily been met by the method described below.

THE PRINCIPLE OF THE METHOD

Attraction of Plane Lamina.—In order to understand the principle of the method, let us choose a coordinate system with the xz-plane as the plane of integration and with the y-axis horizontal and parallel to the strike or axis of the mass configuration. Let the x-axis be horizontal and the z-axis vertical and positive downward. Let the origin be taken as the point at which the gravitational effect of the body will be computed. Since this attraction will be detected by a gravity meter only as an increment Δg to the total earth gravity g, only the z-component of the attraction need be considered and it is this which we seek to compute.

First consider an infinite horizontal plane lamina bounded by the planes z, and $z+dz$ (Fig. 1). Let dS be an element of area of this plane in three-dimensional

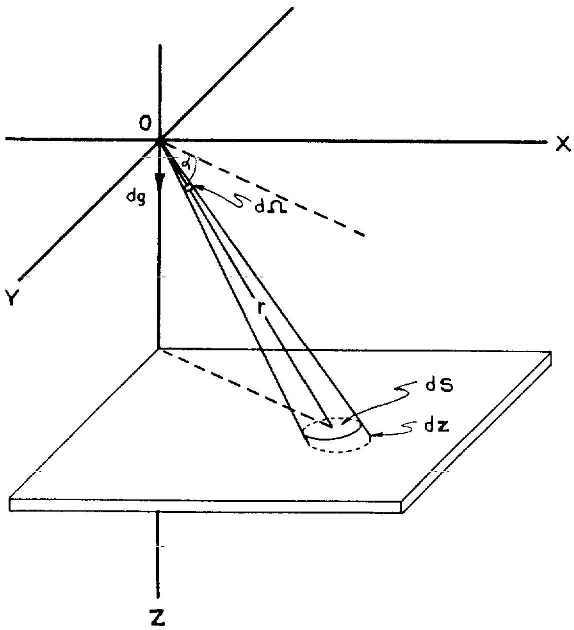

FIG. 1. Attraction of infinite plane lamina.

space and let ρ be the volume density of the element. The vertical component, at the origin, of the attraction due to this element will be

$$dg = \frac{kdm}{r^2} \sin \alpha = \frac{k\rho dz dS}{r^2} \sin \alpha, \tag{1}$$

where k is the constant of gravitation, r the polar distance of the element from the origin, and α the angle of depression of r from the horizon of the station.

But

$$\frac{dS \sin \alpha}{r^2} = d\Omega \tag{2}$$

is the solid angle subtended at the origin by the area dS, so that equation (1) can be written more simply in the form:

$$dg = k\rho dz d\Omega. \tag{3}$$

If we now consider a finite area S of arbitrary shape the attraction at the origin due to the enclosed mass will be

$$g = kdz \int_S \rho d\Omega, \tag{4}$$

and if ρ is constant over S, this simplifies to

$$g = k\rho\Omega dz. \tag{5}$$

Likewise, for a given solid angle Ω, the attraction of the matter enclosed between two horizontal planes z_1 and z_2 will be obtained by integrating equation (5) with respect to z:

$$g = k\Omega \int_{z^1}^{z_2} \rho dz, \tag{6}$$

and again if ρ is constant this becomes,

$$g = k\rho\Omega(z_2 - z_1), \tag{7}$$

which is the contribution to gravity at the origin of the mass contained in the frustrum of a slant cone with vertex at the origin.

Attraction of $d\theta dz$-Solenoid.—Let us next consider the attraction at the origin which will result if we let the element of surface area dS become a narrow linear strip of infinite length parallel to the y-axis. Such a strip will be defined by the area on the plane z = const. lying between two inclined planes which intersect on the y-axis and make with the x-axis angles of θ, and $\theta + d\theta$, respectively (Fig. 2).

As we have seen from equation (5), the attraction of this strip will be proportional to the solid angle subtended by it at the origin. The solid angle $d\Omega$ between two planes intersecting at an angle $d\theta$ will bear the same ratio to the total solid angle which the plane angle $d\theta$ bears to the total plane angle. The total solid angle is the ratio of the surface area of a sphere to the square of its radius, or 4π, and the total plane angle is 2π. Thus

$$\frac{d\Omega}{4\pi} = \frac{d\theta}{2\pi},$$

or

$$d\Omega = 2d\theta. \tag{8}$$

Introducing this into equation (3) then gives for the attraction of such a strip

$$dg = 2k\rho d\theta dz, \tag{9}$$

which we shall regard as the fundamental differential equation of the attraction of a two-dimensional mass. The intersection of the two planes θ and $\theta + d\theta$ with

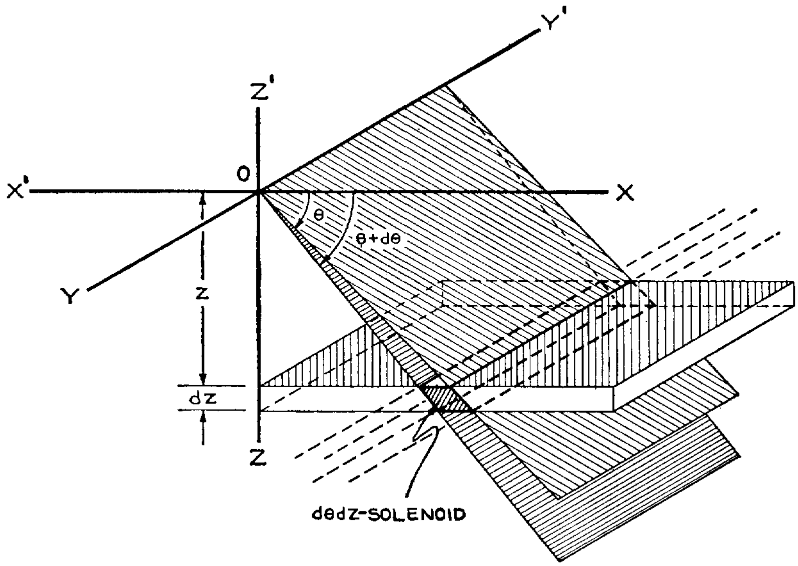

FIG. 2. Illustration of $d\theta dz$-solenoid.

the planes z and $z+dz$ defines an elementary prism or solenoid of infinite length and, in terms of the variables θ and z, of cross section $d\theta dz$. This we shall call the $d\theta dz$-solenoid.

For a finite area in the plane of integration

$$g = 2k \int\int \rho d\theta dz, \tag{10}$$

and when ρ is constant over the area

$$g = 2k\rho(\theta_2 - \theta_1)(z_2 - z_1), \tag{11}$$

independently of the absolute magnitudes of either θ or z.

Areal Integration by Means of $\Delta\theta\Delta z$-solenoids.—Equations (9) and (11) afford at once a simple basis for the computations of the gravimetric effects of two-dimensional masses. The coordinates θ and z are taken as the variables of integration, and the plane of integration is divided by radial lines from the origin, $\theta=$const. with constant spacing $\Delta\theta$, and by horizontal lines $z=$const. with constant spacing Δz, into a mosaic of $\Delta\theta\Delta z$-solenoids.

If the solenoids are chosen small enough that ρ may be considered constant for each (Fig. 3) the contribution to gravity of a single solenoid will be:

$$\Delta g = 2k\rho\Delta\theta\Delta z, \tag{12}$$

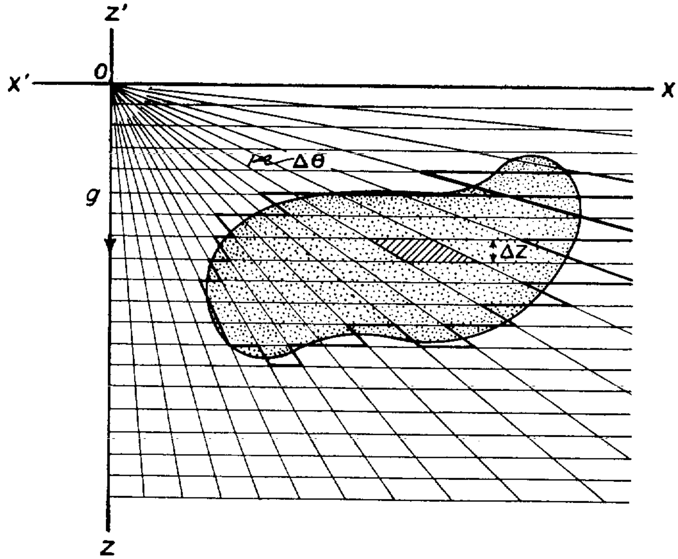

FIG. 3. Calculation of gravitational attraction by areal integration of $d\theta dz$-solenoids.

and the integration over any area will be approximated by

$$g = 2k\sum_{i=1}^{i=n} \rho_i\Delta\theta\Delta z, \tag{13}$$

or, if ρ is constant over the whole area of integration,

$$g = 2k\rho\sum \Delta\theta\Delta z = 2k\rho n\Delta\theta\Delta z, \tag{14}$$

where n is the number of solenoids within the area.

This constitutes a very simple method of computing the effects of two-dimensional masses so long as the area of integration is finite in extent and not too near the horizon. Like most areal integrations, however, it breaks down when these conditions are not satisfied, yet despite this handicap the method is still valuable in that it gives the computer a simple picture of masses having equal gravimetric effects at the station despite variation in size and shape, since for the same density the contribution of each unit solenoid is the same.

Line-integral Method of Integration.—The method just described represents an *areal* integration over the area of cross section of the mass. To avoid some of the difficulties inherent in it, let us now consider the corresponding *linear* integrations around the periphery of the area. Around an elementary solenoid $d\theta dz$ (Fig. 4) bounded by the lines θ and $\theta+d\theta$, and z and $z+dz$, consider the line integral $\oint \theta dz$.

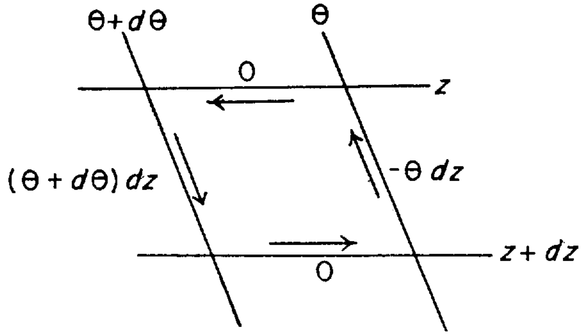

FIG. 4. Line integral about a $d\theta dz$-solenoid.

Starting at the intersection of the lines θ and z, we traverse the circuit in the sense z, $\theta+d\theta$, $z+dz$, θ. In making this circuit dz will be zero on two sides and the resulting integral will be:

$$\begin{aligned}\oint \theta dz &= 0 + (\theta + d\theta)dz + 0 - \theta dz \\ &= d\theta dz. \end{aligned} \tag{15}$$

Traversing the circuit in the opposite direction gives the same result but with a negative sign.

Alternatively, consider the integral $\oint z d\theta$, where the circuit this time is traversed in the sense θ, $z+dz$, $\theta+d\theta$, z.

$$\begin{aligned}\oint z d\theta &= 0 + (z + dz)d\theta + 0 - z d\theta \\ &= + dz d\theta. \end{aligned} \tag{16}$$

Traversing the circuit in the opposite sense again changes the sign.

Over a finite area S we may obtain the $\int\int d\theta dz$ by performing either of the line integrals of equations (15) or (16), and then integrating the results over the area. Thus

$$\int\int_S d\theta dz = \int\int_S \left[\oint \theta dz\right] = \int\int_S \left[\oint z d\theta\right], \tag{17}$$

where the respective line integrals are taken in a positive sense. But as will be seen from Figure 5, when a separate line integral is taken around each of the elementary solenoids of the area, each interior path is traversed twice, once in each direction, whereas the exterior paths are traversed but once, and always in the

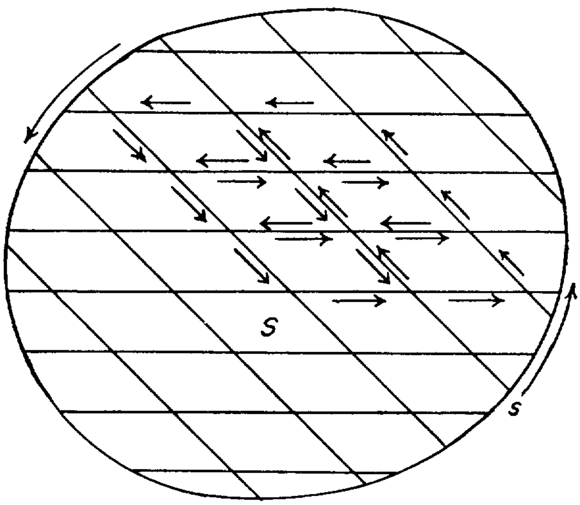

FIG. 5. Conversion of surface integral to line integral.

same sense. Hence, the integrals on all interior paths cancel one another while all those on the exterior paths are cumulative and of like sign. Consequently

$$\iint_S d\theta dz = \oint_s \theta dz = \oint_s z d\theta, \tag{18}$$

where the respective line integrals are each taken in its positive sense around the exterior periphery of the area of integration.

By combining equations (10) and (18) the gravimetric effect of a finite mass of constant density is thus obtainable by either of the integrals:

$$g = 2k\rho \oint \theta dz = 2k\rho \oint z d\theta, \tag{19}$$

taken in the appropriate sense.

Illustrative Example.—To illustrate how the method works let us compute the attraction of the mass bounded by the planes z_1 and z_2 (Fig. 6). Employing the form

$$g = 2k\rho \oint \theta dz,$$

we circumscribe the mass by traversing the surface z_1 from $x = +\infty$ to $x = -\infty$, then descend to the z_2-surface which is traversed from $x = -\infty$ to $x = +\infty$, finally closing the circuit by returning to the z_1-surface at $x = +\infty$.

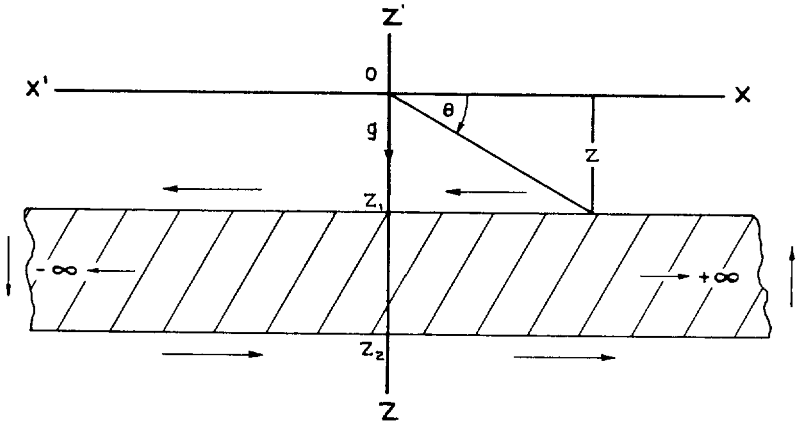

FIG. 6. Calculation of attraction of infinite plate by means of line integrals.

For these four sections of the traverse we have the sum of the four partial integrals:

$$g = 2k\rho\left[\int_{x=+\infty,z=z_1}^{x=-\infty}\theta dz+\int_{z_1,\,x=-\infty}^{z_2}\theta dz+\int_{x=-\infty,z=z_2}^{x=+\infty}\theta dz+\int_{z_2,\,x=+\infty}^{z_1}\theta dz\right]. \quad (20)$$

The first and third integrals vanish since $dz=0$, and the fourth vanishes because $\theta=0$. This leaves only the second integral for which θ assumes the constant value π. Consequently equation (20) reduces by inspection to the familiar result:

$$g = 2\pi k\rho\int_{z_1}^{z_2}dz = 2\pi k\rho(z_2 - z_1).$$

Let us now solve the same problem by means of the equation

$$g = 2k\rho\oint z d\theta.$$

In this case we begin at the point $(x=+\infty,\ z=z_1)$ but traverse the circuit in the opposite direction. For the four sections of the path we find

$$g = 2k\rho\left[\int_{z_1,\,x=-\infty}^{z_2}zd\theta+\int_{x=+\infty\ z=z_2}^{x=-\infty}zd\theta+\int_{z_2,\,x=-\infty}^{z_1}zd\theta+\int_{x=-\infty,z=z_1}^{x=+\infty}zd\theta\right]. \quad (21)$$

In this case the first and third integrals vanish because $d\theta=0$, and the integrands of the second and fourth integrals assume the constant values, respectively, of z_2 and z_1. Hence, the equation reduces to

$$g = 2k\rho\left[z_2\int_{x=+\infty}^{x=-\infty}d\theta + z_1\int_{x=-\infty}^{x=+\infty}d\theta\right]$$
$$= 2k\rho(\pi z_2 - \pi z_1) = 2\pi k\rho(z_2 - z_1).$$

Graphical Evaluation of Line Integrals.—It is thus seen that for any path along which either θ or z is constant the integrals can be evaluated by inspection. In general, however, this special condition is not satisfied, and the paths along which the integrations are to be performed are not capable of simple analytical expression. In these cases the integrals must be evaluated approximately by some form of graphical and numerical calculation. Thus the integrals (19) can be approximated by the summations

$$\left.\begin{aligned} g &= 2k\rho \sum \theta\Delta z, \\ g &= 2k\rho \sum z\Delta\theta. \end{aligned}\right\} \tag{22}$$

By taking constant increments Δz or $\Delta\theta$, these equations simplify to

$$\left.\begin{aligned} g &= 2k\rho\Delta z \sum \theta, \\ g &= 2k\rho\Delta\theta \sum z, \end{aligned}\right\} \tag{23}$$

so that their evaluation consists in determining the value of the integrand (θ or z) for each increment Δz or $\Delta\theta$ around the periphery of the figure, and adding the results, the signs being respectively positive or negative as the increment (Δz or $\Delta\theta$) increases or decreases.

The principle of such a graphical evaluation is illustrated in Figures 7 and 8. The performance of the integration is facilitated if there is superposed upon the figure to be integrated a network of lines $\theta=$const., $z=$const. at constant spacing $\Delta\theta$ and Δz, respectively. From this, for each increment of the variable of integration around the periphery of the figure, the value of the integrand can be read off graphically.

Such an integration is also invariant with respect to distortion as may be seen by imagining the θz-network, as well as the figure to be integrated, to be drawn upon a sheet of thin rubber. Integrations performed before and after giving this sheet an arbitrary distortion would still give the same results since the measuring device is deformed in the same manner as the thing to be measured.

This fact is useful in many geological calculations where the horizontal distances are commonly of the order of 10-fold, or more, times the vertical distances. In such cases it is often convenient to draw cross sections with a large vertical exaggeration. The foregoing integrations are still valid for such cases provided the θz-diagram is drawn with the same vertical exaggeration.

Each of equations (24) consists of two essential factors, the constant $2k\rho$ and the purely geometrical factor $\Delta z \sum\theta$, or $\Delta\theta \sum z$. It is only the latter which is involved in performing an integration. Of the two geometrical components, the angle θ is dimensionless, and z has the dimensions of length. It is convenient, therefore, in making calculations to draw the lines z-constant directly upon the cross section to be integrated and with a spacing in accordance with the vertical scale of the cross section. This eliminates completely any further consideration of the scale of the drawing.

Suppose, for example, that 10 meters has been chosen as the interval for Δz. All that is necessary then is to space the lines, $z=\text{const.}$, 10 meters apart in accordance with the vertical scale of the drawing.

This leaves only the radial lines, $\theta=\text{const.}$, to be accounted for. Since it is often desirable to make computations at successive points along a profile it is convenient to draw the radial lines, $\theta=\text{const.}$, on a transparent template. Since

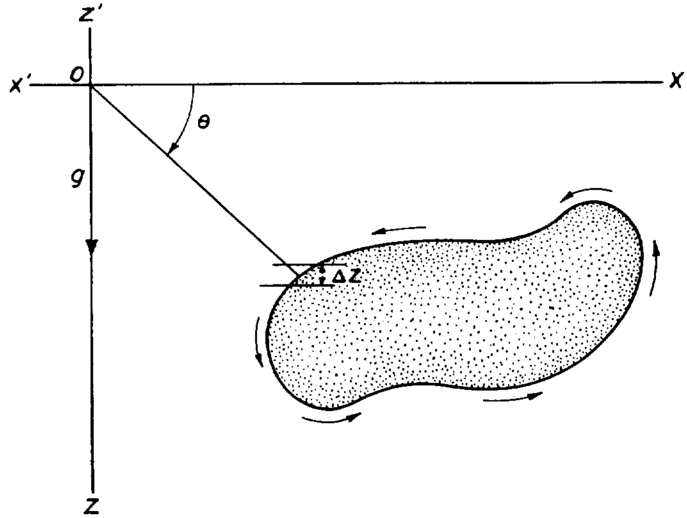

FIG. 7. Geometrical elements involved in evaluating $\oint \theta dz$.

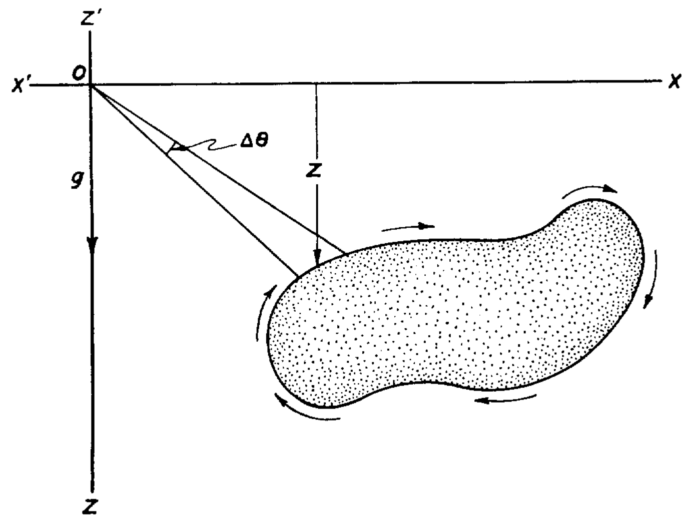

FIG. 8. Geometrical elements involved in evaluating $\oint z d\theta$.

θ is dimensionless such a template will be equally applicable to any cross section of whatever scale provided only that the template has the same vertical exaggeration as the cross section. In practice only a few different vertical distortions are commonly employed, such as, say, 1:1, 1:5, 1:10, etc. Angle templates for each of these can be constructed, and when the proper one of these is placed over the cross section containing both the figure to be integrated and the lines, $z =$ const., nothing more remains to be done except to read off the numbers and talley the results.

Either an areal or a linear integration may be employed, depending upon the nature of the figure.

Optimum Size of $\Delta\theta\Delta z$—Solenoid.—A final word may be said about the optimum size for a $\Delta\theta\Delta z$-solenoid. This may well vary with the nature of the problem. One useful choice might be of a unit solenoid which would produce an increment of gravity of 0.01 milligal (10^{-5} gal) when filled with a material whose density is 1 gm/cm^3. Solving equation (12) with these numerical data gives

$$\Delta\theta\Delta z = \frac{\Delta g}{2k\rho} = \frac{10^{-5}}{2 \times 6.67 \times 10^{-8}} = 75.0 \text{ cm.}$$

Thus if $\Delta\theta$ were 1 radian, z would be 75 cm.; if it were 0.1 radian, Δz would be 7.5 meters, etc.

A convenient interval for $\Delta\theta$ would appear to be about 0.05 radians (ca. 2.5°) for which Δz would be 15 meters (45.8 ft.), which should suffice for many geological calculations. In general, however, the choice of the optimum size for the $\Delta\theta\Delta z$-solenoid would depend upon the problem under consideration and should be made to suit one's individual needs.