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Power Tools

By STEVEN STROGATZ

If you were an avid television watcher in the 1980s, you may remember a clever show called "Moonlighting." Known for its snappy dialogue and the romantic chemistry between its co-stars, it featured Cybill Shepherd and Bruce Willis as a couple of wisecracking private detectives named Maddie Hayes and David Addison. While investigating one particularly tough case, David asks a coroner's assistant for his best guess about possible suspects. "Beats me," says the assistant. "But you know what I don't understand?" To which David replies, "Logarithms?" Then, reacting to Maddie's look: "What? You understood those?"

(See website to watch clip.)

That pretty well sums up how many people feel about logarithms. Their peculiar name is just part of their image problem. Most folks never use them again after high school, at least not consciously, and are oblivious to the logarithms hiding behind the scenes of their daily lives.

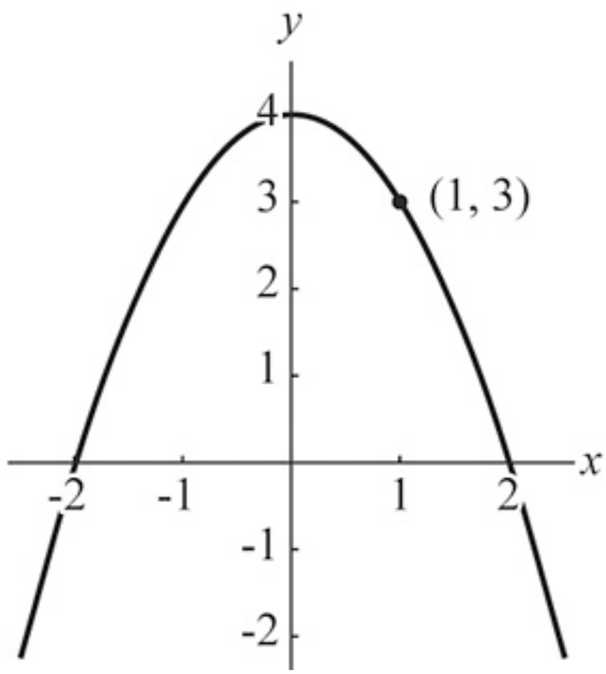
The same is true of many of the other functions discussed in algebra II and pre-calculus. Power functions, exponential functions — what was the point of all that? My goal in this week's column is to help you appreciate the function of all those functions, even if you never have occasion to press their buttons on your calculator.

A mathematician needs functions for the same reason that a builder needs hammers and drills. Tools transform things. So do functions. In fact, mathematicians often refer to them as "transformations" because of this. But instead of wood or steel, functions pound away on numbers and shapes and, sometimes, even on other functions.

To show you what I mean, let's plot the graph of the equation

$$y = 4 - x^2.$$

You may remember how this sort of activity goes: you draw a picture of the *xy* plane with the *x*-axis running horizontally and the *y*-axis vertically. Then for each *x* you compute the corresponding *y* and plot them together as a single point in the *xy* plane. For example, when *x* is 1, the equation says *y* equals 4 minus 1 squared, which is 4 minus 1, or 3. So $(x,y) = (1, 3)$ is a point on the graph. After calculating and plotting a few more points, the following picture emerges.



The droopy shape of the curve is due to the action of mathematical pliers. In the equation for y, the function that transforms x into x^2 behaves a lot like the common tool for bending and pulling things. When it's applied to every point on a piece of the x-axis (which you could visualize as a straight piece of wire), the pliers bend and elongate that piece into the downward-curving arch shown above.

And what role does the 4 play in the equation $y = 4 - x^2$? It acts like a nail for hanging a picture on a wall. It lifts the bent wire arch up by 4 units. Since it raises all points by the same amount, it's known as a "constant function."

This example illustrates the dual nature of functions. On the one hand, they're tools: the x^2 bends the piece of the x-axis and the 4 lifts it. On the other hand, they're building blocks: the 4 and the $-x^2$ can be regarded as component parts of a more complicated function, $4 - x^2$, just as wires, batteries and transistors are component parts of a radio.

Once you start to look at things this way, you'll notice functions everywhere. The arching curve above — technically known as a "parabola"— is the signature of the squaring function x^2 operating behind the scenes. Look for it when you're taking a sip from a water fountain or watching a basketball arc toward the hoop. And if you ever have a few minutes to spare on a layover in Detroit's International Airport, be sure to stop by the Delta terminal to enjoy the world's most breathtaking parabolas at play:

Parabolas and constants are associated with a wider class of functions — "power functions" of the form x^n, in which a variable x is raised to a fixed power n. For a parabola, $n = 2$; for a constant, $n = 0$.

Changing the value of n yields other handy tools. For example, raising x to the first power ($n = 1$) gives a function that works like a ramp, a steady incline of growth or decay. It's called a "linear function" because its xy graph is a line. If you leave a bucket out in a steady rain, the water collecting at the bottom rises linearly in time.

Another useful tool is the inverse square function $1/x^2$, corresponding to the case $n = -2$. It's good for describing how waves and forces attenuate as they spread out in three dimensions — for instance, how a sound softens as it moves away from its source.

Power functions like these are the building blocks that scientists and engineers use to describe growth and decay in their mildest forms.

But when you need mathematical dynamite, it's time to unpack the exponential functions. They describe all sorts of explosive growth, from nuclear chain reactions to the proliferation of bacteria in a Petri dish. The most familiar example is the function 10^x, in which 10 is raised to the power x. Make sure not to confuse this with the earlier power functions. Here the exponent (the power x) is a variable, and the base (the number 10) is a constant — whereas in a power function like x^2, it's the other way around. This switch makes a huge difference. Exponential growth is almost unimaginably rapid.

That's why it's so hard to fold a piece of paper in half more than 7 or 8 times. Each folding approximately doubles the thickness of the wad, causing it to grow exponentially. Meanwhile, the wad's length shrinks in half every time, and thus *decreases* exponentially fast. For a standard sheet of notebook paper, after 7 folds the wad becomes thicker than it is long, so it can't be folded again. It's not a matter of the folder's strength; for a sheet to be considered legitimately folded n times, the resulting wad is required to have 2^n layers in a straight line, and this can't happen if the wad is thicker than it is long.

The challenge was thought to be impossible until Britney Gallivan, then a junior in high school, solved it in 2002. She began by deriving a formula

$$L = \frac{\pi T}{6}\left(2^n + 4\right)\left(2^n - 1\right)$$

that predicted the maximum number of times, n, that paper of a given thickness T and length L could be folded in one direction. Notice the forbidding presence of the exponential function 2^n in two places — once to account for the doubling of the wad's thickness at each fold, and another time to account for the halving of its length.

Using her formula, Britney concluded that she would need to use a special roll of toilet paper nearly three quarters of a mile long. In January 2002, she went to a shopping mall in her hometown of Pomona, Calif., and unrolled the paper. Seven hours later, and with the help of her parents, she smashed the world record by folding the paper in half 12 times!

In theory, exponential growth is also supposed to grace your bank account. If your money grows at an annual interest rate of r, after one year it will be worth $(1 + r)$ times your original deposit; after two years, $(1 + r)$ squared; and after x years, $(1 + r)^x$ times your initial deposit. Thus the miracle of compounding that we so often hear about is caused by exponential growth in action.

Which brings back to logarithms. We need them because it's always useful to have tools that can undo one another. Just as every office worker needs both a stapler and a staple remover, every mathematician needs exponential functions *and* logarithms. They're "inverses." This means that if you type a number x into your calculator, and then punch the 10^x button followed by the $\log x$ button, you'll get back to the number you started with.

Logarithms are compressors. They're ideal for taking numbers that vary over a wide range and squeezing them together so they become more manageable. For instance, 100 and 100 million differ a million-fold, a gulf that most of us find incomprehensible. But their logarithms differ only fourfold (they are 2 and 8, because $100 = 10^2$ and 100 million = 10^8). In conversation, we all use a crude version of logarithmic shorthand when we refer to any salary between $100,000 and $999,999 as being "six figures." That "six" is roughly the logarithm of these salaries, which in fact span the range from 5 to 6.

As impressive as all these functions may be, a mathematician's toolbox can only do so much — which is why I still haven't assembled my Ikea bookcases.

NOTES:

1. The excerpt from "Moonlighting" is from the episode "In God We Strongly Suspect." It originally aired on Feb. 11, 1986, during the show's second season.
2. Will Hoffman and Derek Paul Boyle have filmed an intriguing video of the parabolas all around us in the everyday world (along with their exponential cousins, curves called "catenaries," so-named for the shape of hanging chains). Full disclosure: the filmmakers say this video was inspired by a story I told on an episode of

RadioLab.
3. For simplicity, I've referred to expressions like x^2 as functions, through to be more precise I should speak of "the function that maps x into x^2." I hope this sort of abbreviation won't cause confusion, since we've all seen it on calculator buttons.
4. For the story of Britney Gallivan's adventures in paper folding, see: ***Gallivan, B. C. "How to Fold Paper in Half Twelve Times: An 'Impossible Challenge' Solved and Explained." Pomona, CA: Historical Society of Pomona Valley, 2002.*** *For a journalist's account, aimed at children, see* ***Ivars Peterson, "Champion paper-folder," Muse (July/August 2004), p. 33.*** *The Mythbusters have also attempted to replicate Britney's experiment on their television show.*
5. For evidence that our innate number sense is logarithmic, see: ***Stanislas Dehaene, Véronique Izard, Elizabeth Spelke, and Pierre Pica, "Log or linear? Distinct intuitions of the number scale in Western and Amazonian indigene cultures," Science, Vol. 320 (2008), p. 1217.*** *Popular accounts of this study are available at ScienceDaily and in this episode of RadioLab.*

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