A simple solution to tunnel or well discharge under constant drawdown

Pierre Perrochet

Abstract A simple analytical formula is developed to calculate transient discharge inflow rates into a tunnel or a well under constant drawdown. The agreement with the classical, but cumbersome diffusion-equation-based solution of Jacob and Lohman is excellent throughout the range of dimensionless times. By using only a straightforward logarithmic function, this explicit solution may therefore be used with great computational benefits in practice, and also when further mathematical manipulations such as differentiation or integration are required.

Resumen Una fórmula analítica sencilla fue desarrollada para calcular el grado de descarga transitoria hacia un túnel, o un pozo, bajo condición de un abatimiento constante. Existe una concordancia excelente, desde el comienzo hasta el final, en el rango de los tiempos adimensionales, con la solución clásica pero complicada, de Jacob y Lohman, basada esta última en la ecuación de difusión. Solamente mediante el uso de una función logarítmica simple, esta solución explícita puede por tanto ser usada en la práctica con grandes ventajas computacionales, y también cuando se necesitan manipulaciones matemáticas adicionales, tales como diferenciación o integración.

Résumé On a développé une formule analytique simple pour le calcul du flux infiltré en régime transitoire dans un tunnel ou un puits en supposant le rabattement constante. Les résultats sont en accord avec la solution plus compliquée de Jacob-Lohman de l'équation de diffusion, sur tout l'intervalle de temps adimensionel considéré. En utilisant une fonction logarithmique ce solution explicite peut être utilisée sans un grand effort de calcul dans la pratique courante, ainsi que dans les situations où il est nécessaire à dériver ou intégrer l'expression du rabattement.

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Classical Solution

Confined horizontal flow towards a fully penetrating well in a semi-infinite, homogeneous aquifer of constant thickness is classically described by the radial form of the diffusion equation. Assuming uniform hydrostatic initial heads and a sudden, constant drawdown at the well, an analytical expression for the resulting transient discharge rates was first published by Jacob and Lohman (1952) who applied the heat conduction solution of Smith (1937) to groundwater dynamics.

Consider the radial form of the diffusion equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rT\frac{\partial s}{\partial r}\right) = S\frac{\partial s}{\partial t}, r_o \leqslant r \leqslant \infty, 0 \leqslant t \leqslant \infty \tag{1}$$

with the initial and boundary conditions

$$s(r,0) = 0, s(r_o,t) = s_o, s(\infty,t) = 0 \tag{2}$$

where the symbols stand for aquifer transmissivity (T), storage coefficient (S), time (t), radial coordinate (r), well radius (r_o), drawdown (s) and drawdown at the well (s_o).

The above problem was analysed by Smith who resorted to Green's functions and integral transforms but could only find a solution for $\partial s/\partial r$ at the origin. Using this result, Jacob and Lohman deduced the flow rate at the origin (i.e. into the well) as

$$Q(t) = -2\pi r_o T\frac{\partial s}{\partial r}(r_o,t) = 2\pi T s_o G(\alpha) \tag{3}$$

with

$$G(\alpha) = \frac{4\alpha}{\pi}\int_0^\infty u\,\mathrm{e}^{--\alpha u^2}\left\{\frac{\pi}{2} + \tan^{--1}\left(\frac{Y_o(u)}{J_o(u)}\right)\right\}du, \alpha = \frac{Tt}{Sr_o^2} \tag{4}$$

where α is dimensionless time, J_o and Y_o are first and second kind zero-order Bessel functions respectively, and u is a dummy variable.

The above solution has become the reference formula used to evaluate the transient discharge rates at artesian wells or tunnels (e.g. de Marsily 1981; Maréchal and Perrochet 2003). However, the complicated nature of the function $G(\alpha)$ makes its continuous evaluation or analytical manipulation difficult. Consequently, an alternative solution is proposed below which overcomes these practical and mathematical drawbacks.

Proposed Solution

Based on the classical working assumptions enforced in two-dimensional, confined groundwater dynamics, the problem defined in Eqs. (1) and (2) is that of transient Darcian radial flow through a semi-infinite, homogeneous aquifer towards a well under constant drawdown.

Using the standard linear diffusion equation in this context implies that the drawdown specified at the origin has an instantaneous, infinitely small effect throughout the aquifer, and that this effect is strictly nil only at $r=\infty$. Following the approach used successfully by Perrochet and Musy (1992) for one-dimensional unconfined drainage, an alternative way of defining the problem is to consider that the effect of the specified drawdown strictly vanishes beyond a no-flow moving boundary located at a time-dependent distance $r=R(t)$.

As schematised in Fig. 1, the domain of interest is then $r_o \leq r < R(t)$, and the boundary conditions in Eq. (2) need to be modified and complemented by

$$s(r_o, t) = s_o, \frac{\partial s}{\partial r}(R(t), t) = 0, s(R(t), t) = 0 \tag{5}$$

Integrating Eq. (1) over this domain yields the global time-dependent quantities

$$Q(t) = \frac{\partial}{\partial t} \int_{r_o}^{R(t)} 2\pi\, r S\, s(r,t) dr = \frac{\partial V(t)}{\partial t} \tag{6}$$

where $Q(t)$ is the discharge at the well and $V(t)$ is the cumulative volume of extracted water.

Assuming now that the drawdown propagation can be treated as successive steady-state snapshots of the function $s(r, t)$ over the distance $R(t)$, one can simply replace the right-hand side of Eq. (1) by a uniform time-dependent source term and consider the equation to solve as

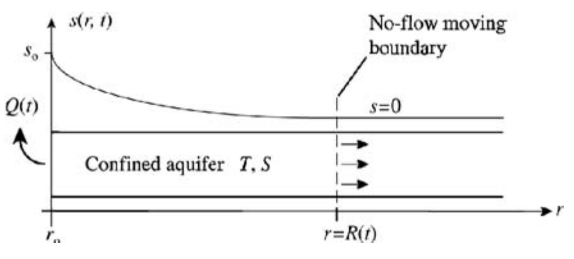

Fig. 1 Sketch of one-dimensional radial flow towards a well with a no-flow moving boundary at $r=R(t)$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rT\frac{\partial s}{\partial r}\right) = i(t), r_o \leqslant r < R(t) \tag{7}$$

The two constants of integration as well as the function $i(t)$ are easily obtained accounting for the three boundary conditions in Eq. (5), and the solution of Eq. (7) is

$$s(r,t) = s_o\left(1 - \frac{2R(t)^2 \ln(r/r_o) - r^2 + r_o^2}{2R(t)^2 \ln(R(t)/r_o) - R(t)^2 + r_o^2}\right) \tag{8}$$

Introducing this function in the central member of Eqs. (3) and (6), the global quantities $Q(t)$ and $V(t)$ can be expressed as

$$Q = 2\pi T s_o \left(\ln\left(x^{\frac{x^2}{x^2-1}}\right) - 1/2\right)^{-1} \tag{9}$$

$$V = \pi r_o^2 S s_o \left(x^2 - 4\ln\left(x^{\frac{x^2}{x^2-1}}\right) + 1\right)\left(4\ln\left(x^{\frac{x^2}{x^2-1}}\right) - 2\right)^{-1} \tag{10}$$

where $x=R/r_o$, and with the bracketed time symbol omitted for simplicity.

Considering Eq. (6) and the variation of V with that of R, one can write

$$Q = \frac{\partial V}{\partial R}\frac{\partial R}{\partial t} \tag{11}$$

which, after substitution of Q and $\partial V/\partial R$ from Eqs. (9) and (10), separation of variables, re-arrangement of terms and integration, becomes

$$\frac{Tt}{Sr_o^2} = \int_1^x \frac{\ln\left(u^{\frac{u^2+1}{u^2-1}}\right) - 1}{4\ln\left(u^{\frac{u^2}{u^2-1}}\right) - 2}\, u\, du \tag{12}$$

At this stage the right-hand side of the above equation could not be fully integrated but, by polynomial analysis, the following approximation was found excellent:

$$\frac{Tt}{Sr_o^2} \cong \frac{1}{\pi e}\left(x^{\frac{x^2}{x^2-1}} - \sqrt{e}\right)^2 \tag{13}$$

Rewriting Eq. (13) as

$$x^{\frac{x^2}{x^2-1}} \cong \sqrt{e}\left(1 + \sqrt{\frac{\pi T t}{S r_o^2}}\right) \tag{14}$$

and substituting into Eq. (9) for the discharge rate at the well yields

$$Q \cong \frac{2\pi T s_o}{\ln\left(1 + \sqrt{\frac{\pi T t}{S r_o^2}}\right)} \tag{15}$$

Comparing Eq. (15) with Eq. (3), and using the dimensionless time defined in Eq. (4) results in

$$G(\alpha) \cong \frac{1}{\ln(1 + \sqrt{\pi\alpha})} \tag{16}$$

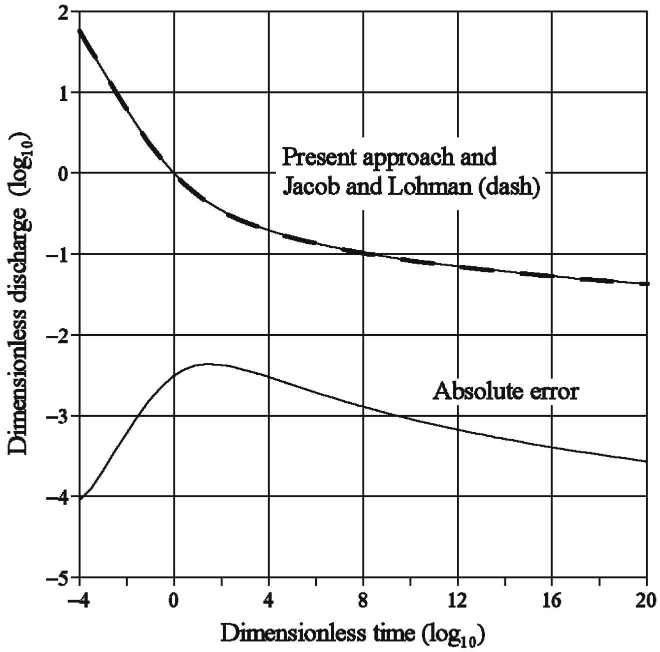

Fig. 2 Plot of the functions $G(\alpha)$, $\ln(1+\sqrt{\pi\alpha})^{-1}$ and absolute error

A graphical comparison between the functions (16) and (4) and the absolute error (difference between the two functions) is shown in Fig. 2 for dimensionless times in the range $10^{-4}<\alpha<10^{20}$. $G(\alpha)$ was computed in this case in the Laplace-transform space and reverted into the time domain by a standard inversion technique. The match between the two curves is excellent throughout this fairly large range, and errors are maximum in the range $10^{1}<\alpha<10^{2}$ at about 5×10^{-3}. The errors vanish for very small and very large times, for which the righthand side of Eq. (16) takes the same asymptotic trends as the original function $G(\alpha)$.

Equation (15) may therefore replace Eq. (3) to predict transient discharge rates at a well or a tunnel under confined conditions and constant drawdown. The conditions of application of Eq. (15) are of course restricted to those of Eq. (3), themselves dictated by the assumptions enforced in classical well hydraulics in infinite and homogeneous aquifers. However, one explicit insight is gained here as to the time during which Eqs. (15) or (3) may hold.

This time must be smaller than the time needed for the drawdown perturbation to reach a known aquifer boundary at a given distance L (i.e. impervious layer, surface water body, etc.). Equations (15) or (3) are therefore valid to predict transient discharge rates as long as the no-flow moving boundary at the distance $R(t)$ from the well or the tunnel has not reached any system boundary. For relatively large times, $R(t)>>r_o$ ($x>>1$), Eq. (14) indicates that this distance (radius of influence) scales with $\sqrt{\pi e\alpha}$, and that the time of validity $t_{\lim}$, such as $R(t_{\lim})=L$, can be evaluated by

$$t_{\lim}=\frac{Sr_o^2}{\pi T}\left(\frac{L}{r_o\sqrt{e}}-1\right)^2\cong\frac{SL^2}{\pi eT} \tag{17}$$

For larger times, Eqs. (15) or (3) must not be used anymore because the flow conditions depart from those defined in Eqs. (5) or (2). Discharge rates should then be predicted by other means such as image-based analytical solutions or numerical simulators.

Concluding Remarks

The alternative solution presented in this note strongly suggests the replacement of the well function $G(\alpha)$ of Jacob and Lohman by $\ln(1+\sqrt{\pi\alpha})^{-1}$ for any α value. This has two major benefits.

Firstly, this new solution allows straightforward, pocket-calculator evaluations with excellent practical accuracy over the whole range of dimensionless times. Secondly, this substitution makes further numerical or analytical operations much easier, such as time differentiation, integration or convolution.

Moreover, the comparison with the classical, diffusion-equation solution for the confined radial case demonstrates the validity of the approach and the idea that a number of other flow problems could be analysed in a similar manner. Accurate, but greatly simplified new solutions could therefore be obtained for typical confined or unconfined flow configurations.

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