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Theory and Methodology

Optimal decision making in a maintenance service operation

D.N.P. Murthy *, E. Asgharizadeh

Department of Mechanical Engineering, The University of Queensland, St.Lucia, Q4072, Australia

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Abstract

In recent years, there has been a dramatic increase in the out sourcing of equipment maintenance with the maintenance being carried out by an external agent rather than in-house. Often, the agent offers more than one option and the owners of equipment (customers) are faced with the problem of selecting the optimal option. The optimal choice depends on their attitude to risk and the parameters of the different options. The agent needs to take these issues into account in the optimal selection of the parameters for the different options and this requires a game theoretic formulation. The papers deals with one such model formulation to determine the agent's optimal strategy with regards the pricing structure, the number of customers to service and the number of service channels.

Keywords: Maintenance; Service contract; Game theory; Optimal pricing

1. Introduction

Industrial equipments are unreliable in the sense that they deteriorate with age and/or usage and ultimately fail. The failures can have a significant impact on the business performance. Maintenance actions are used to control failures and to restore a failed equipment back to operational status. Many different approaches to maintenance (e.g., Reliability Centered Maintenance, Tero-technology, Integrated Life Cycle) have been proposed in the literature and a large number of models have been developed to determine the optimal maintenance strategies. There are several books which deal with the different approaches (see, for example, Kelly, 1984; Blanchard, 1981; Mourbay, 1994) and there are several excellent review papers dealing with different models for determining the optimal maintenance (see, for example, Pierskalla and Voelker, 1976; Sherif and Smith, 1981; Thomas, 1986). These deal mainly with maintenance being carried out in-house.

Complex equipment requires specialist tools and personnel to carry out repairs after failures. Often it is uneconomical for owners of such equipment to have the specialist tools and personnel in house. In such

* Corresponding author. E-mail: murthy@mech.uq.edu.au.

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situations, it is more economical to out source the maintenance (preventive and corrective) of such equipment. In this case, the maintenance is carried out by an external agent and the owner can be viewed as a customer of the agent for the maintenance service. Henceforth, we shall use the terms "agent" and "customer" to denote the provider and the recipient of the maintenance service respectively. We confine our attention to the case of a single agent and multiple customers.

The literature on "service" is also very vast. Bulk of it deals with various aspects of service in a qualitative manner (see, for example, Bleul and Bender, 1980; Murdick et al., 1990; Norman, 1991; Blumberg, 1991). Papers which deal with the modelling and analysis of maintenance service (provided by an external agent) are few and we review them later in the section.

In this paper, we confine our attention to an industrial equipment which generates revenue to the customer, over the life of the equipment, when it is in working state and no revenue when it is in failed state. Hence, the duration for which the equipment is in failed state is critical for the customer. The product is unreliable in the sense that the failures occur in an uncertain manner.

The agent offers the following two options to the customer to maintain the equipment.

Option A_1 (*Service Contract*): For a fixed price of P, the agent agrees to repair all failures over $[0, L)$ at no cost. L is the life of the useful life of the equipment. If the failed equipment is not returned to working state within a period τ subsequent to the failure, the agent incurs a penalty. If Y denote the time to return a failed unit back to working state, then the penalty incurred by the agent is $\max\{0, \alpha(Y-\tau)\}$.

Option A_2 (*No service Contract*): In this case, whenever the equipment fails, the agent repairs it at a cost of C_s per repair. There is no penalty if the time to repair exceeds τ. Under this option, the total cost of repair (over the life L) to a customer is a random variable since the number of failures is uncertain.

The customer's choice between these two options is influenced by the price structure and the attitude of the customer to risk. We assume that the customer is risk averse and that the customer's decision is based on maximizing an expected utility function. Note that if the price structure is such that the expected utility is negative, the customer might prefer the following option.

Option A_0: The customer does not buy the product and hence needs no maintenance service from the external agent.

As a result, the customer has to choose the optimal option A^* from the set given $\{A_1, A_2, A_0\}$.

When there is only a single customer, the agent has to decide on the optimal pricing structure (P in Option A_1 and C_S in Option A_2) taking into account the customers response. Murthy and Asgharizadeh (1995, 1996) developed a Stackelberg game theoretic model formulation to obtain the optimal pricing strategy with the agent as the leader and the customer as the follower. The model assumes exponential failure times so that there is no need for preventive maintenance actions and corrective maintenance actions involve minimal repair (see Murthy, 1991). This is appropriate for complex electronic equipment. They give a complete analytical characterization of the optimal strategy for the agent and discuss the effect of model parameter variations on the agent's optimal strategy.

In Asgharizadeh and Murthy (1996), the authors extend their earlier model to include more than one customer but with only a single service channel. This implies that when an equipment fails, its repair cannot commence immediately if there is one or more failed equipment waiting to be repaired. This has an impact on the revenue generation for customers as well as on the agent's profit. In this case, the number of customers to service is an extra decision variable (in addition to the pricing structure) which the agent must select optimally. Using results from queuing theory, the authors give a complete characterization of the optimal strategy for the agent using a game theoretic formulation.

With multiple customers and a single service channel, the mean waiting time for failed equipment increases with the numbers of customers that the agent services. One way of reducing this is to have more than one service channel so that more than one failed equipment can be repaired at any given time. However, this results in additional (set up) costs to the agent and in turn affects the total profit. In this paper we include this in the model formulation so that the decision variables for the agent are the price structure, the number of customers to service and the number of service channels.

The outline of the paper is as follows. In Section 2, we give the details of the model formulation. Section 3 deals with the model analysis to characterize the optimal strategies and we illustrate it with a numerical example in Section 4. Finally, in Section 5, we briefly discuss some extensions to the model studied in this paper.

2. Model formulation

In this section we give the details of the model formulation.

2.1. Equipment failures

Each customer owns a single unit which is used to generate revenue for the customer. The revenue generated is R per unit time when the equipment is in working state and no income when in failed state. The useful life of all units is L and the purchase price is C_b per unit.

All units are statistically similar in terms of reliability. The time to first failure is given by an exponential distribution with mean equal to $1/\lambda$. This implies that the failure rate is λ. After failure, the failed unit is minimally repaired (see Murthy, 1991). Under minimal repair, the failure rate after repair is the same as that before failure. As a result, failure time for a repaired item is also exponentially distributed with mean $1/\lambda$. As mentioned earlier, this characterization is appropriate for complex electronic equipment.

2.2. Maintenance of equipment

Because the failure rate is assumed constant, preventive maintenance is not appropriate and so need not be considered. Hence, the only maintenance provided by the agent is corrective maintenance. The time to repair a failed equipment is exponentially distributed with mean $1/\mu$. The above assumption is necessary to make the analysis tractable. The cost (labour + material) to the agent for carrying out a repair is C_m.

With the purchase of equipment, each customer has to choose between options A_1 and A_2 for maintenance since only the agent can carry out the corrective maintenance. On the other hand, if owning and operating the equipment is unprofitable, the optimal option for the customer might be option A_0. We will assume that the agent selects the price structure so that customers will always choose between options A_1 and A_2 and never forced to choose option A_0.

Let M denote the number of customers serviced by the agent using S service channels.

2.3. Customer's decision problem

We assume that all M customers are identical in their attitude to risk. The optimal choice for each is based on maximizing the expected utility function. We assume that it is given by

$$U(\omega) = (1 - e^{-\beta\omega})/\beta, \tag{1}$$

where $U(\omega)$ is the utility associated with a wealth of ω. The advantage of this utility function is that the initial wealth is of no importance. (See Murthy and Padmanabhan (1993) for further discussion.) Note that this captures the attitude to risk. $\beta = 0$ corresponds to the risk neutral case with $U(\omega) = \omega$ and the risk aversion increases with β increasing.

For customer j $(1 \leqslant j \leqslant M)$, let the number of failures over [0, L) be N_j. Let X_{ji} denote the time to failure after $(i-1)$th $(2 \leqslant i \leqslant N_j)$ repair. Let $\tilde{X}_j$ denote the time for which the unit is in operational state at the end of its useful life subsequent to being put into operational state after the last repair. Note that it is zero if the unit is in failed state when it reaches the end of its useful life. Let Y_{ji} $(1 \leqslant i \leqslant N_j)$ denote the time taken to make the equipment operational after the ith failure. This time includes the waiting time and the time to repair. X_{j1} is the time to failure for the item purchased by customer j. Let $\omega(\mathrm{A}_k)$ denote the return to jth customer under option A_k, $1 \leqslant k \leqslant 2$. Then, it is easily seen that

$$\omega(\mathrm{A}_1) = R\left(\sum_{i=1}^{N_j} X_{ji} + \tilde{X}_j\right) + \alpha\left[\sum_{i=1}^{n_j} \max\{0, (Y_{ji} - \tau)\}\right] - C_\mathrm{b} - P, \tag{2}$$

$$\omega(\mathrm{A}_2) = R\left(\sum_{i=1}^{N_j} X_{ji} + \tilde{X}_j\right) - C_\mathrm{b} - C_\mathrm{s} N_j. \tag{3}$$

Finally, under option A_0, we have

$$\omega(\mathrm{A}_0) = 0. \tag{4}$$

Note that X_{ji}, Y_{ji} and $\tilde{X}_j$ are random variables. They are affected by M and S since these have an impact on the waiting times. Since all M customers are similar in their attitude to risk and since all M units are statistically identical, the expected utility under action A_k $(1 \leqslant k \leqslant 2)$ is the same for all M customers. It is a function of the decision variables (P, C_s, M, S) of the agent. Let $U(\mathrm{A}_k; P, C_\mathrm{s}, M, S)$ denote the utility to customer j under action A_k, $1 \leqslant k \leqslant 2$.

Each customer chooses between the three options (A_0, A_1 and A_2) to maximize the expected utility $E[U(\mathrm{A}_k; P, C_\mathrm{s}, M, S)]$. If the expected utility under A_1 and A_2 are negative, then the optimal strategy for the customers is A_0 and in this case $E[U(\mathrm{A}_0; P, C_\mathrm{s}, M, S)] = 0$ since $\omega(\mathrm{A}_0) = 0$ with probability one.

2.4. Agent decision problem

The agent's profit depends on the decisions of the customers and the number of service channels. The cost of setting up and operating S $(S \leqslant M)$ channels is a nonlinear function of S. Since all M customers are the same, they choose the same option for maintenance. Let the agent's profit under option A_k be denoted by $\pi(P, C_\mathrm{s}, M, S; \mathrm{A}_k)$ for $0 \leqslant k \leqslant 2$.

When the customers choose option A_1, the agent's profit is given by

$$\pi(P, C_\mathrm{s}, M, S; \mathrm{A}_1) = \sum_{j=1}^{M}\left[P - C_\mathrm{m} N_j - \alpha\left(\sum_{i=1}^{N_j} \max\{0, (Y_{ji} - \tau)\}\right)\right] - C_0 S - C_1 S^2, \tag{5}$$

where the first term in the square bracket represents the revenue generated, the second term is the cost of repairing failures for customer j and, the final term represents the penalty paid to customer j, $1 \leqslant j \leqslant M$. The last two terms represent the setup cost associated with the S service channels. Note that it is quadratic in S.

Similarly, when the customers choose option A_2, the agent's profit is given by

$$\pi(P, C_\mathrm{s}, M, S; \mathrm{A}_2) = \sum_{j=1}^{M} [C_\mathrm{s} - C_\mathrm{m}] N_j - C_0 S - C_1 S^2. \tag{6}$$

Finally, when the customers choose option A_0, then the agent's profit is given by

$$\pi(P, C_\mathrm{s}, M, S; \mathrm{A}_0) = 0. \tag{7}$$

since the agent has no customers and there is no need to invest in a service facility.

The agent chooses his decision to maximize the expected profits. Note that they are dependent on the actions of the customers.

2.5. Stackelberg game formulation

As mentioned earlier, the agent needs to take into account the optimal actions of the customers in order to determine the optimal choice of P, C_s, M, S – the decision variables under the agent's control. The optimal actions of customers depend on these decision variables of the agent. As a consequence, a game theoretic approach is the most natural and appropriate to determine the agent's optimal strategies.

In this game formulation, the agent is the leader and the customers are the followers. The customers select the optimal option A^* (from the set A_0, A_1 and A_2) to maximize their expected utility. The agent first derives the optimal option A^* as a function of P, C_S, M and S (the decision variables of the agent) which maximizes the customers expected utility. (This is also known as customer response function.) The agent's optimal decisions (P^*, C_S^*, M^* and S^*) are then obtained by maximizing the expected profit using the A^* $(= A^*(P, C_S, M, S))$ in the expression for the expected profit.

Our formulation models the case where the agent has the proprietary knowledge needed for maintenance and is a monopolist. As a result, the pricing decisions are made by the agent and the agent can be viewed as a leader and the customers as followers. In other words, the agent's optimal strategy is obtained using a Stackleberg game formulation (see Basar and Olsder (1982) for more on game theory).

In carrying out the analysis, we assume complete information in the sense that customers know the product failure rate (λ) and the actions of the agent and that the agent knows the risk parameter (β). We also assume that both parties are rational and interested in maximizing their own objectives (expected utility in the case of customers and expected profit in the case of the agent). Relaxing some these assumptions is discussed later in the paper.

3. Model analysis

We assume that failed units are repaired on a first come first served basis. In addition, we make the following simplifying assumptions:

Assumption 1: L is sufficiently large in relation to mean time between failures so that one can use steady state results for the distribution for Y_{ji}.

Assumption 2: $E(Y_{ji}) \ll E(X_{ji}) = (1/\lambda)$. This implies that the mean total (waiting + repair) time is very small in relation to mean time to failure. This is also a valid assumption for well designed equipment. As a result, the total down time of equipment for each customer is small in relation to L so that

$$R\left[\sum_{i=1}^{N_j} X_{ji} + \tilde{X}_j - \sum_{i=1}^{N_j} Y_{ji}\right] \approx R\left[\sum_{i=1}^{N_j} X_{ji} + \tilde{X}_j\right] \approx RL.$$

Also, this implies that N_j, $1 \leqslant j \leqslant M$, are Poisson distributed with mean λL.

3.1. Steady state distribution for Y_{ji}

Note that the model formulation is similar to a Markovian queue with a finite population (M) and finite number of servers (S). The arrival rate is given by

$$\lambda_k = \begin{cases} (M-K)\lambda & \text{for} \quad 0 \leqslant K \leqslant M, \\ 0 & \text{for} \quad k > M, \end{cases} \tag{8}$$

the departure rate by

$$\mu_k = \begin{cases} k\mu & \text{for} \quad 0 \leqslant k \leqslant S, \\ S\mu & \text{for} \quad k > S, \end{cases} \tag{9}$$

and the queue operates on a first come first served rule. Let Y denote the total (waiting + service) time in the system and $f(y)$ the density function for Y. The density function for Y_{ji} is also given by $f(y)$ since Y and Y_{ji} are identically distributed.

The steady state distribution for the above Markovian queue model is given by (see White et al., 1975, pp. 113–120)

$$f(y) = \mathrm{e}^{-\mu y}\sum_{k=0}^{S-1}\hat{P}_k + \sum_{k=1}^{M-S}\hat{P}_{k+s-1}\mu(S\mu)^k\left[\{\mathrm{e}^{-\mu y}/(S\mu-\mu)^k\} - \sum_{j=1}^{k}[\{y^{k-j}\mathrm{e}^{-s\mu y}\}/\{(k-j)!(S\mu-\mu)^j\}]\right] \tag{10}$$

with $\hat{P}_k$ given by

$$\hat{P}_k = \{(M-k)P_k\}\Big/\left\{\sum_{k=0}^{M}(M-k)P_k\right\} \tag{11}$$

and

$$P_k = \begin{cases} (\lambda/\mu)^k\{M!/(M-k)!k!\}P_0 & \text{for} \quad k = 0,1,\ldots,S-1, \\ (\lambda/S\mu)^k(S^S/S!)\ \{M!/(M-k)!\}P_0 & \text{for} \quad k = S, S+1,\ldots,M, \\ 0 & \text{for} \quad k > M, \end{cases} \tag{12}$$

and P_0 given by

$$P_0 = \left[\sum_{k=0}^{S-1}(\lambda/\mu)^k\{(M)!/(M-k)!k!\} + \sum_{k=s}^{M-1}(\lambda/S\mu)^k(S^S/S!)\{M!/(M-k)!\}\right]^{-1}. \tag{13}$$

The expected value of Y_{ji} is given by

$$E[Y_{ji}] = 1/\mu + \sum_{k=s}^{M-1}P_k(k-S+1)\Big/S\mu. \tag{14}$$

3.2. Customer's optimal strategy

For customer j, let the number of times the agent incurs penalty be $\tilde{N}_j(\leqslant N_j)$. The expected utility, $E[U(\mathrm{A}_1;\ P,\ C_\mathrm{s},\ M,\ S)]$, needs to be evaluated by carrying out the expectation over N_j, $\tilde{N}_j$ and the Y_{ji}'s.

Define I_{ji}, $1 \leqslant j \leqslant M$ and $1 \leqslant i \leqslant N_j$, as follows:

$I_{ji} = 1$ if $Y_{ji} > \tau$ and $= 0$ otherwise.

Using this, and the approximation of Assumption 2, Eq. (2) can be written as

$$\omega(\mathrm{A}_1) = RL - C_b - P + \alpha\sum_{i=1}^{N_j}[I_{ji}\ (Y_{ji}-\tau)] \tag{15}$$

and the number of times the agent incurs penalty with customer j, $\tilde{N}_j$, is given by

$$\tilde{N}_j = \sum_{i=1}^{N_j} I_{ji}. \tag{16}$$

Then $U(\mathrm{A}_1;\ P,\ C_\mathrm{s},\ M,\ S)$ is given by Eq. (1) with $\omega(\mathrm{A}_1)$ given by Eq. (15). Note that this is a random variable since N_j, $\tilde{N}_j$ and the Y_{ji}'s are random variables.The expected utility, $E[U(\mathrm{A}_1;\ P,\ C_\mathrm{s},\ M,\ S)]$, is obtained by carrying out the expectation over these random variables. We do this using a three stage conditional approach as indicated below.

$$E[U(\mathrm{A}_1; P, C_\mathrm{s}, M, S)] = E\,E[U(\mathrm{A}_1; P, C_\mathrm{s}, M, S \mid N_j)] = E\,E\,E[U(\mathrm{A}_1; P, C_s, M, S \mid \tilde{N}_j \mid N_j)], \tag{17}$$

where the expectations are done over $Y_{ji}|\tilde{N}_j, Nj$; $\tilde{N}_j|N_j$ and N_j, respectively.

From Eq. (1) and Eq. (15) we have

$$E[U(\mathrm{A}_1; P, C_s, M, S \mid \tilde{N}_j, N_j)] = (1/\beta)\ \{1 - [\mathrm{e}^{-\beta(RL - C_b - P)}]\} E\left[\prod_{i=1}^{N_j} \mathrm{e}^{-\{\beta\alpha\ I_{ji}(Y_{ji} - \tau)\}} \mid \tilde{N}_j,\ N_j\right]. \tag{18}$$

Note that only $\tilde{N}_j$ of the Y_{ji}'s are $>\tau$ and the rest are $<\tau$. Also, the Y_{ji}'s are statistically independent. As a result, we have

$$E\left[\prod_{i=1}^{N_j} \mathrm{e}^{-\{\beta\alpha\ I_{ji}(Y_{ji} - \tau)\}} \mid \tilde{N}_j,\ N_j\right] = \prod_{i=1}^{\tilde{N}_j} \int_\tau^\infty \mathrm{e}^{-\beta\alpha(y-\tau)}\ [f(y)/(1 - F(y)]\ \mathrm{d}y$$

and this can be rewritten as

$$E\left[\prod_{i=1}^{N_j} \mathrm{e}^{-\{\beta\alpha\ I_{ji}(Y_i - \tau)\}} \mid \tilde{N}_j,\ N_j\right] = \left[\int_\tau^\infty \mathrm{e}^{-\beta\alpha(y-\tau)}\ [f(y)/(1 - F(y)]\ \mathrm{d}y\right]^{\tilde{N}_j}. \tag{19}$$

Define

$$\psi = (1/\beta)\ \mathrm{e}^{-\beta(RL - C_\mathrm{b} - P)}.$$

Using this, Eq. (18) can be rewritten as

$$E[U(\mathrm{A}_1; P, C_\mathrm{s}, M, S \mid \tilde{N}j, N_j)] = (1/\beta) - \psi\left[\int_\tau^\infty \mathrm{e}^{-\beta\alpha(y-\tau)}\ [f(y)/(1 - F(y)]\ \mathrm{d}y\right]^{\tilde{N}_j}. \tag{20}$$

Note that $\tilde{N}_j$, conditional on N_j, is binomially distributed random variable with parameter $F(\tau)$ since the agent incurs a penalty when $Y_{ji} > \tau$. As a result,

$$P\left\{\tilde{N}_j = j \mid N_j\right\} = \binom{N_j}{j}\ \{1 - F(\tau)\}^j\ \{F(\tau)\}^{(N_j - j)}. \tag{21}$$

Using this in the unconditioning process, we have

$$E[U(\mathrm{A}_1; P, C_s, M, S \mid N_j)] = \sum_{j=1}^{N_j} E[U(\mathrm{A}_1; P, C_\mathrm{s}, M, S \mid \tilde{N}_j = j, N_j]\ \Pr\left\{\tilde{N}_j = j, N_j\right\}. \tag{22}$$

Using Eq. (20) and Eq. (21) in Eq. (22) we have

$$E[U(\mathrm{A}_1; P, C_s, M, S \mid N_j)] = (1/\beta) - \psi\left\{\int_{\tau}^{\infty} \mathrm{e}^{-\beta\alpha(y-\tau)} f(y)\mathrm{d}y + F(\tau)\right\}^{N_j}. \tag{23}$$

Finally,

$$E[U(\mathrm{A}_1; P, C_s, M, S)] = \sum_{n=0}^{\infty} E[U(\mathrm{A}_1; P, C_s, M, S \mid N_j)] \Pr\{N_j = n\}. \tag{24}$$

Since N_j is Poisson distributed with mean λ_L, we have

$$\Pr\{N_j = n\} = \left\{(\lambda L)^n \mathrm{e}^{-(\lambda L)^n}\right\}/n!\,. \tag{25}$$

Using Eq. (23) and Eq. (25) in Eq. (24), and after some simplification, we have

$$E[U(\mathrm{A}_1; P, C_s, M, S)] = (1/\beta)\left\{1 - \mathrm{e}^{-\beta(RL-C_b-P)} + \lambda L\left[\int_{\tau}^{\infty} \mathrm{e}^{-\beta\alpha(y-\tau)} f(y)\mathrm{d}y + F(\tau) - 1\right]\right\}. \tag{26}$$

Using a similar approach, we have

$$E[U(\mathrm{A}_2; P, C_s, M, S)] = (1/\beta)\left\{1 - \mathrm{e}^{-\beta(RL-C_b)} - \lambda L(1 - \mathrm{e}^{\beta C_s})\right\}. \tag{27}$$

Finally, note that

$$E[U(\mathrm{A}_0; P, C_s, M, S)] = 0. \tag{28}$$

For a given (P, C_s, M, S), a comparison between the three expected utilities will indicate which action is the optimal one. For a fixed M and S, the optimal customer strategy A^* ($=\mathrm{A}^*(P, C_S)$) is characterized by three regions (Ω_i, $0 \leqslant i \leqslant 2$,) in the ($P$, C_s) plane as shown in Fig. 1. In Ω_0, $\mathrm{A}^* = \mathrm{A}_0$; in Ω_1, $\mathrm{A}^* = \mathrm{A}_1$ and in Ω_2, $\mathrm{A}^* = \mathrm{A}_2$. The curve ($\Gamma$) separating Ω_1 and Ω_2 is given by

$$P = (\lambda L/\beta)\left(\mathrm{e}^{\beta C_s} - \xi(M, S)\right) \tag{29}$$

with $\xi(M, S)$ given by

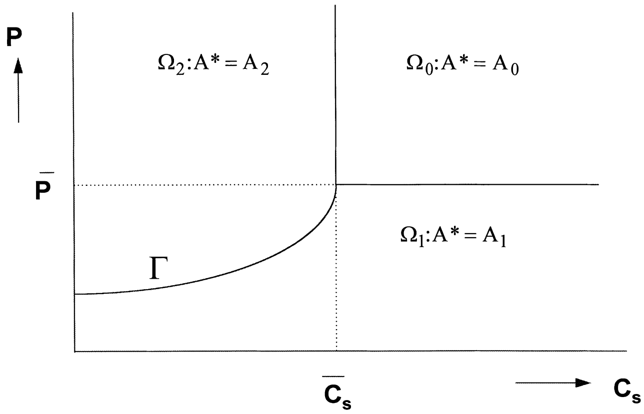

Fig. 1. Customer's optimal strategies.

$$\xi(M,S) = \sum_{k=0}^{S-1} P_k[(\beta\alpha+\mu) - \beta\alpha e^{-\mu\tau}]/\mu(\beta\alpha+\mu) + \sum_{k=1}^{M-S} P_{k+S-1}\,\mu(S\mu)^k \Bigg[\Big\{ [(\beta\alpha+\mu) - \beta\alpha e^{-\mu\tau}]/\mu(\beta\alpha+\mu)(S\mu-\mu)^k \Big\} + \sum_{r=1}^{k} 1/(S\mu-\mu)^r \Big\{ \sum_{j=0}^{k-r} \Big[\tau^{k-r-j} e^{-s\mu\tau}/((k-r-j)!)[1/(\beta\alpha+S\mu)^{j+1} - 1/(S\mu)^{j+1}\Big] + 1/(S\mu)^{k-r+1} \Big\} \Bigg] \quad (30)$$

and $\overline{P}(M,S)$ and $\overline{C_s}(M,S)$ are given by

$$\overline{P}(M,S) = RL - C_b - (\lambda L/\beta)(1 - \xi(M,S)) \quad (31)$$

and

$$\overline{C_s}(M,S) = (1/\beta)\ln\{1 + \beta(RL - C_b)/(\lambda L)\}. \quad (32)$$

The implications of this is as follows. For a fixed M and S, if the agent chooses the price structure such that $P > \overline{P}(M,S)$ and $C_S > \overline{C_s}(M,S)$, then the optimal strategy for the customers is A_0. Buying the equipment and operating it under either A_1 or A_2 will result in a negative expected utility. Hence, for the agent to stay in business, it is essential that the agent ensures that either $P \leqslant \overline{P}(M,S)$ or $C_S \leqslant \overline{C_s}(S,M)$.

For a fixed M, as S increases, the curve Γ moves downwards. $\overline{P}(M,S)$ decreases as S increases. The reason for this is as follows. S increasing results in shorter waiting time for repair and hence less expected penalty cost. As a result, $\overline{P}$ must decrease in order for the service contract (option A_1) to be attractive or else the customers will opt for option A_2. Note that $\overline{C_s}(M,S)$ does not depend on either M or S.

3.3. Agent's optimal strategy

We derive the agent's optimal strategy (P^*, C_S^*, M^* and S^*) using a two-stage approach. In the first stage, we fix M and S. The optimal $A^*(P, C_S)$ [1] depends on P and C_S as indicated in the previous section. The optimal values, $P^*(M, S)$ and $C_s^*(M, S)$ are obtained by maximizing $E[\pi(P, C_s, M, S; A^*(P, C_S))]$. We give a complete analytical characterization of $P^*(M, S)$ and $C_s^*(M, S)$ later in the section. In the second stage, M^* and S^* are obtained by maximizing $E[\pi(P^*(M, S), C_s^*(M, S), M, S; A^*]$. This is an integer optimization problem. Once this is done, P^* and C_S^* are given by $P^* = P^*(M^*, S^*)$ and $C_S^* = C_s^*(M^*, S^*)$.

It can be easily shown that for a fixed M and S, the agent must select either $P^*(M,S) > \overline{P}(M,S)$ and $C_s^*(M,S) = \overline{C_s}(M,S)$ or $P^* = \overline{P}(M,S)$ and $C_s^*(M,S) > \overline{C_s}(M,S)$ to maximize the expected profit when customers choose their strategy to maximize the utility function. As a result, the expected agent's profit is given by

$$E[\pi(P^*, C_s^*, M, S; A^*)] = \begin{cases} M(\overline{P} - \lambda L(C_m - \xi)) - (C_0 S + C_1 S^2), & \text{when } C_s > \overline{C_s},\ P = \overline{P}, \\ M\lambda L[\overline{C_s} - C_m] - (C_0 S + C_1 S^2), & \text{when } C_s = \overline{C_s},\ P > \overline{P}, \\ 0, & \text{when } C_s > \overline{C_s},\ P > \overline{P}, \end{cases} \quad (33)$$

where for notational ease we have suppressed the arguments (M, S) and denote $\xi(M,S)$, $\overline{P}(M,S)$ and $\overline{C_s}(M,S)$ simply as ξ, $\overline{P}$ and $\overline{C_s}$.

[1] Often, we shall omit the argument for notational ease and denote the customers optimal action $A^*(P, C_s)$ as A^*.

When $P = \overline{P}(M, S)$ and $C_s > \overline{C_s}(M, S)$, the customer's optimal choice A^* is A_1; when $P > \overline{P}(M, S)$ and $C_s = \overline{C_s}(M, S)$, then A^* is A_2 and finally when $P > \overline{P}(M, S)$ and $C_s > \overline{C_s}(M, S)$, then A^* is A_0. In the first two cases, the customer's expected utility is zero and there is no consumer surplus. This implies that the agent, as a monopolist, extracts the maximum amount from the customer. Charging any more, i.e., $P > \overline{P}(M, S)$ and $C_s > \overline{C_s}(M, S)$ results in customers choosing A_0 instead of A_1 or A_2.

As a result, for a fixed M and S, the agent optimal action is the choice between:
(i) $P^*(M, S) > \overline{P}(M, S)$ and $C_s^*(M, S) = \overline{C_s}(M, S)$ and
(ii) $P^*(M, S) = \overline{P}(M, S)$ and $C_s^*(M, S) > \overline{C_s}(M, S)$.
The one which yields a higher value yields the optimal choice. Once this is done, the next step is to proceed to the second stage to obtain M^* and S^* and this is done as follows:

For a fixed $S (S = 1, 2, \ldots)$, let $M^*(S)$ denote the value of M which maximizes

$$E[\pi(P^*(M, S), C_s^*(M, S), M, S; A^*(P^*(M, S), C_S^*(M, S)))].$$

One can obtain this by an exhaustive search. Using this, S^* is given by the value of S which maximizes

$$E[\pi(P^*(M^*(S), S), C_s^*(M^*(S), S), M^*(S), S; A^*(P^*(M^*(S), S), C_S^*(M^*(S), S)))]$$

and this can also be obtained by an exhaustive search. From this, we have $M^* = M^*(S^*)$, $P^* = P^*(M^*, S^*)$ and $C_s^* = C_s^*(M^*, S^*)$.

3.4. Sensitivity analysis

The two important model parameters are β (the customer risk parameter) and λ (the failure rate of the equipment). We discuss the effect of the variations of these parameters on the optimal strategies of customers and agent.

3.4.1. Effect of β variations

For a given M and S, from Eqs. (31) and (32), it can be easily shown that $d\overline{P}(M, S)/d\beta < 0$ and $d\overline{C}s(M, S)/d\beta < 0$. This implies that both $\overline{P}(M, S)$ and $\overline{C}_s(M, S)$ decrease as β increases. Also, the curve Γ moves upward as β increases. As a result, with β increasing, the horizontal line (corresponding to $P = \overline{P}(M, S)$) moves down and the vertical line (corresponding to $C_s = \overline{C_s}(M, S)$) moves to the left. Fig. 2 shows the plots for two values of β with $\beta_2 > \beta_1$.

Let $\overline{P}_i(M, S)$ and $\overline{C}_{si}(M, S)$ denote the $\overline{P}(M, S)$ and $\overline{C}_s(M, S)$ for β_1, $1 \leqslant i \leqslant 2$. For $0 < C_s < C_{s2}(M, S)$, we see that for a fixed C_s, a more risk averse customer is willing to pay a higher price for the service contract relative to a low risk customer. Similarly, for a fixed $P(> \overline{P}_1(M, S))$ a low risk customer is willing to buy the equipment and get each failure repaired individually as long as $C_s < \overline{C}_{s1}(M, S)$ whereas a high risk customer will opt for A_0 when $C_s > \overline{C}_{s2}(M, S)$. These results are as expected and agree intuitively with the anticipated behavior of customers with increasing risk aversion. Unfortunately, it is not possible to give any analytical results to indicate the effect of β on the optimal decisions of the agent.

When $\beta = 0$, both customers and the agent are risk neutral. In this case, the agent's optimal profits (for a given M and S) under options A_1 and A_2 are the same, given by

$$E[\pi(P^*, C_s^*, M, S; A^* = A_1)] = E[\pi(P^*, C_s^*, M, S; A^* = A_2)] = M(RL - C_b - \lambda L C_m) - (C_0 S + C_1 S^2).$$

The waiting time to repair does not have any impact on the agent's profit. The reason for this is that the agent recovers the higher penalty payment through an increase in the value for $P^*(M, S)$ and the customers do not mind this as they get compensated through the penalty payments made by the agent. Note that the optimal profit is a linear function of M.

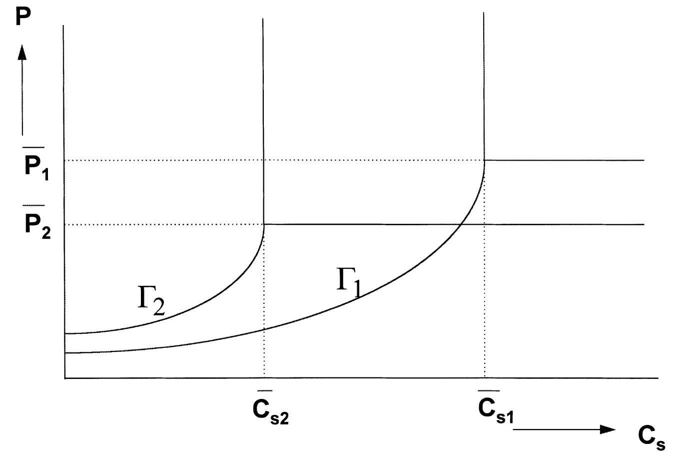

Fig. 2. Effect of β variations ($\beta_2 > \beta_1$).

The above expression is not valid for large M. This is because our derivation is based on Assumption 2 and when M is large, the total (waiting + service) time is no longer small relative to the mean time between failures.

3.4.2. Effect of λ variations

For a given M and S, from Eqs. (31) and (32), it can be easily shown that $\mathrm{d}\overline{P}(M,S)/\mathrm{d}\lambda > 0$ and $\mathrm{d}\overline{C}_s(M,S)/\mathrm{d}\lambda < 0$. This implies that $\overline{P}(M,S)$ increases and $\overline{C}_s(M,S)$ decreases as λ increases. Also, the curve Γ moves upward as λ increases. As a result, with λ increasing, the horizontal line (corresponding to $P = \overline{P}(M,S)$) moves up and the vertical line (corresponding to $C_s = \overline{C}_s(M,S)$) moves to the left. The consequence of this is that the region Ω_1 increases and Ω_2 shrinks as λ increases.

4. Numerical example

We consider the following nominal values for the model parameters: $\lambda = 0.0010$/h, $\mu = 0.02$/h, $\alpha = 0.06$/h (10^3 \$), $\beta = 0.1$, $\tau = 70$ h, $C_b = 300$ (10^3 \$), $L = 40{,}000$ h, $R = 0.015$/h (10^3 \$), $C_0 = 300$ (10^3 \$) and $C_1 = 150$ (10^3 \$).

From Eq. (32) we have $\overline{C}_s(M,S) = 5.596$ (10^3 \$). As mentioned earlier, this does not depend on M and S. In contrast, $\overline{P}(M,S)$ (given by Eq. (31)) changes with M and S. Table 1 shows $\overline{P}(M,S)$ for a range of M and S values. For a given M, as S increases, $\overline{P}(M,S)$ decreases and reaches $\overline{P}(1,\ 1)$ when $S = M$ and does not change for $S > M$. The reason for this is that when $S = M$, there is a dedicated channel for each customer. For $S > M$, $(S - M)$ channels are always idle and hence $S \leqslant M$. For a given S, as M increases, $\overline{P}(M,S)$ also increases for reasons discussed earlier.

Table 2 shows $E[\pi(P^*(M,S), C_s^*(M,S), S, M; \mathrm{A}^*)]$ for a range of M and S values. Consider $S = 3$. For $M \leqslant 10$, we see that the agent's expected profit is negative for both options A_1 and A_2 since the costs of carrying out the maintenance exceed the revenue generated. In this case the optimal strategy for the agent is not to provide the maintenance service. As a result, the profit is zero. This situation is indicated by (#) next to 0 in the table. For $M > 80$, the expected profits is zero since the customers strategy is option A_0. This situation is indicated by (@) next to 0 in the table.

Table 1
$\bar{P}(M, S)$ for a range of M and S

M	S				
	1	2	3	4	5
1	322.763	322.763	322.763	322.763	322.763
5	333.870	323.035	322.763	322.763	322.763
10	356.521	324.516	322.763	322.763	322.763
20	458.869	333.244	323.842	322.879	322.763
40	693.255	419.910	335.241	324.842	323.146
60	699.964	645.177	398.802	335.537	325.467
80	699.999	712.500	586.314	385.146	335.284
100	699.999	1916.629	681.756	537.009	375.481

Table 2
$E[\pi(P^*(M,S),\ C_s^*(M,S),\ S,\ M;\ A^*(P^*(M,S),\ C_s^*(M,S))]$ for a range of $(S,\ M)$ combinations

M	S				
	1	2	3	4	5
1	0 (#)	0 (#)	0 (#)	0 (#)	0 (#)
5	179.288	0 (#)	0 (#)	0 (#)	0 (#)
10	616.685	125.737	0 (#)	0 (#)	0 (#)
20	0 (@)	1382.455	406.608	0 (#)	0 (#)
40	0 (@)	1377.526	2839.017	1700.819	72.200
60	0 (@)	0 (@)	3120.212	4120.271	2689.870
80	0 (@)	0 (@)	0 (@)	4556.230	5677.582
100	0 (@)	0 (@)	0 (@)	0 (@)	375.481

(#): Agent's optimal strategy is not to provide maintenance; (@): Agent's expected profit is zero since $A^* = A_0$.

The implication of this is that for a given S, there is a $M_1(S)$ and a $M_2(S)$ ($>M_1(S)$) such that the agent's expected profit is zero for $M < M_1(S)$ and $M > M_2(S)$. There is an optimal $M^*(S)$ ($M_1(S) \leqslant M^*(S) \leqslant M_2(S)$) which can be obtained by an exhaustive search.

Table 3 shows $M^*(S)$ for S varying from 1 to 9. As can be seen, $M^*(S)$ increases with S. This is to be expected since the set up cost for carrying out maintenance increases with S and hence the agent needs to have more customers to recover the costs. Note that if $M > M^*(S)$, then the agent's expected profit decreases due to increase in the penalty payments exceeding the extra revenue generated by the increase in the number of customers.

Table 3
$M^*(S)$ and $E[\pi(P^*(M^*(S),S),\ C_s^*(M(S),S),\ M^*(S),S;A^*)]$ for $1 \leqslant S \leqslant 9$

S	$M^*(S)$	$E[\pi(P^*(M^*(S),\ S),\ C_s^*(M^*(S),\ S),\ M^*(S);\ A^*)]$
1	13	807.895
2	33	2342.905
3	53	3776.180
4	73	4990.787
5	94	5955.664
6	115	6651.235
7	136	7068.687
8	157	7202.365
9	159	7044.438

Also shown in Table 3 are $E[\pi(P^*(M^*(S),S), C_s^*(M^*(S),S), S, M^*(S); A^*)]$. As can be seen, $S^* = 8$ and M^* $(= M(S^*)) = 157$ maximizes the agent's expected profits. As a result, the optimal pricing strategy is given by $C_s^* = C_s^*(M^*, S^*) > 5.596$ (10^3 \$) and $P^* = P^*(M^*, S^*) = 352.428$ (10^3 \$) and the agent's optimal expected profit is \$7202.365 (10^3 \$).

4.1. Effect of β variations

The effect of β variations on the optimal customer and agent strategies are shown in Table 4 for β varying from 0 to 0.65. Both the optimal number of customers (M^*) and the optimal number of service channels (S^*) decrease as β increases. The optimal pricing strategy is given by $P^* = \overline{P}$ and $C^* > \overline{C_S}$ so that the customers optimal action is to choose the service contract (option A_1). As can be seen from the table, both $\overline{P}$ and $\overline{C}_S$ decrease as β increases. Finally, the agents's optimal expected profit $E[\pi(P^*, C_s^*, M^*, S^*; A^*)]$ (denoted by $E(\pi)^*$ in the table for notational ease) decreases as β increases.

4.2. Effect of λ variations

The effect of λ variations on the optimal customer and agent strategies are shown in Table 5 for λ varying from 0.0010 to 0.0014. Both the optimal number of customers (M^*) and the optimal number of service channels (S^*) decrease as λ increases. The optimal pricing strategy is given by $P^* = \overline{P}$ and $C^* > \overline{C}_S$ so that the customers optimal action is to choose the service contract (option A_1). As can be seen from the table, $\overline{P}$ increases and $\overline{C}_S$ decreases as λ increases. Finally, the agent's optimal expected profit $E[\pi(P^*, C_s^*, M^*, S^*; A^*)]$ (denoted by $E(\pi)^*$ in the table for notational ease) decreases as λ increases. Note that when $\lambda = 0.0014$, the agent's expected profit, with optimal M^*, S^*, P^* and C_s^*, is negative. In this case, the optimal strategy for the agent is not to provide the maintenance service.

Table 4
M^*, $S^*$and $E[\pi]^*$ versus β

	β					
	0.0	0.10	0.20	0.40	0.60	0.65
M^*	159	157	131	126	124	103
S^*	8	8	7	7	7	6
P	375.441	352.428	336.426	322.757	316.930	315.969
$\bar{C}_s$	7.5	5.596	4.581	3.466	2.841	2.724
$E[\pi]^*$	10260.00	7202.365	5733.930	4503.020	3891.510	3781.035

Table 5
$M^*(S)$, $S^*$and $E[\pi]^*$ versus λ

	λ				
	0.0010	0.0011	0.0012	0.0013	0.0014
M^*	157	104	59	38	7
S^*	8	6	4	3	1
$\bar{P}$	352.428	356.883	359.540	362.948	371.570
$\bar{C}_s$	5.596	5.198	4.855	4.554	4.289
$E[\pi]^*$	7202.364	3668.096	1542.552	393.353	–136.137

Finally, it is worth noting that for this numerical example, the variations in λ have a more dramatic impact on the optimal strategy than variations in β.

5. Comments and extensions

In this paper we have developed a model and carried out the analysis to determine the optimal pricing strategy (P, C_s), the number of customers to service (M) and the number of service channels (S) for a monopolist service agent providing the maintenance service.

The model analysis is based on two simplifying assumptions. Assumption 2 requires the expected total (waiting + service) time ($E[Y_{ji}]$) $\ll$ the mean time to failure ($E[X_{ji}]$). When M becomes very large this assumption is no longer valid and in this case

$$R\left[\sum_{i=0}^{N_j} X_{ji} + \tilde{X}_j - \sum_{i=0}^{N_j} Y_{ji}\right] \ncong RL.$$

Also, N_j is no longer given by a Poisson distribution. Rather, it is given by an alternating renewal process with X_{ji} exponentially distributed and Y_{ji} distributed according to $f(y)$ given by (10). This makes the analysis intractable.

Assumption 1 allowed us to use the steady state distribution for Y_{ji}. For a more exact analysis, one would need to use transient distribution and in this case it is impossible to obtain any analytical results. One would then need to use simulation approach to determine the optimal strategies.

The model assumes that the failure times and repair times are exponentially distributed. This allows one to use results from Markov queues to derive some analytical results. If one or both of the distributions are non-exponential, the analysis is too complex and intractable.

If the failure distribution is general (as opposed to exponential) and failures are repaired minimally (with repair times being negligible), then the failures can be modelled by a non-homogeneous Poisson process with intensity function equal to the failure rate of the equipment. In this case, we can derive analytical expressions for the customer's expected utility and the expressions would be more complex.

The model assumes that both parties (agent and customers) have perfect information regarding the model parameters. In other words, the agent knows the true value of β (customers risk parameter) and the customers know the true value of λ (failure rate of equipment). In real life this is seldom true. One way of modelling this is to treat β and λ as random variables so that their true values are unknown. (Murthy and Asgharizadeh (1996), deal with this issue in the case of a single customer and for a simpler characterization where β and λ are modelled as binary valued random variables.)

In our model, the customers are homogeneous in their attitude to risk. The model can be extended to include non-homogeneous customer population with different attitude to risk. In the simplest case, customers are divided into K different groups with each group comprising of people with similar attitude to risk. This allows the agent to have different service contracts for different groups.

Also, in our model, the agent offers two options: (i) a service contract to fix all failures for a fixed price and, (ii) the customer pays for each repair. A simple extension is as follows: The agent offers two types of service contracts with differing penalty refunds in case the equipment is not made operational within a time period τ. Obviously, the pricing would need to be different.

Finally, we have confined our attention to a monopolistic agent. An interesting extension is where there are two or more service agents providing either identical or differing service contracts. This introduces additional issues such as competition and collusion.

As can be seen from the above discussion, there is a need for further research into the modelling and analysis of maintenance service contracts.

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