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# Service Contracts: A Stochastic Model

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**Abstract**—There is a growing trend to outsource maintenance where equipment failures are rectified by an external agent under a service contract. The agent's profit is influenced by many factors—the terms of the contract, equipment reliability, and the number of customers being serviced. The paper develops a stochastic model to study the impact of these on the agent's expected profit and the agent's optimal strategies using a game theoretic formulation. 

**Keywords**—Maintenance, Equipment reliability, Service contract, Expected profit, Game theory.

## 1. INTRODUCTION

Complex equipment requires specialist tools and personnel to carry out repairs when equipment fails. Often it is uneconomical for the owner of the equipment to have such specialist tools and personnel in house. In such situations, it is more economical to outsource the maintenance (preventive and failure) of such equipment. We consider the case where the repair is done by an external agent who provides the repair service. In this case, the owner can be viewed as a customer of the agent for such repair service. Henceforth, we shall use the terms "agent" and "customer" to denote the service provider and the owner of the equipment.

The equipment generates revenue to the customer, over the life of the equipment, when it is in working state, and no revenue when it is in failed state. Hence, the duration for which the equipment is in failed state is critical for the customer. The options offered by the agent have an impact on the customer's decision whether to buy the equipment and the type of contract for repair, should the customer decide on buying the equipment for revenue generation purposes.

Murthy and Ashgarizadeh [1] developed a game theoretic model to characterise the optimal strategies for the customer and the agent. The model deals with a single customer and a single agent. The failure times and repair times are assumed to be exponentially distributed. The service agent offers two options:

(i) to rectify all failures, offer the life of the equipment for a fixed price ($P$) along with a penalty should the repair be not completed within the specified time, and
(ii) to rectify each failure at a fixed price ($C_s$), but with no penalty terms included.

The customer chooses the optimal decision to maximise an expected utility function, and as a result, the optimal decision is a function of $P$ and $C_s$. The decision variables of the agent are $P$ and $C_s$, and this is selected so as to maximise the expected profit taking into account the optimal decision of the customer. In other words, the optimal decisions are obtained using a Stakelberg

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game formulation with the agent as the leader and the customer as the follower. Murthy and Ashgarizadeh [1] give a complete analytical characterisation of the optimal decisions for both the customer and the agent.

In this paper, we extend the model to include multiple customers. This implies that when equipment fails, its repair cannot commence immediately if there are one or more failed pieces of equipment needing repair. This has an impact on the revenue generation for customers and the agent's profit. The agent must choose the optimal number of customers he should service and the optimal pricing strategy. Similarly, each customer must choose between service contract or no service contract with the purchase of equipment. If the total expected revenue is negative, the optimal choice for the customer is not to purchase the equipment. We examine the optimal strategies for customers as well as the agent in a game theoretic setting similar to our earlier paper.

The outline of the paper is as follows. In Section 2, we give the details of the model formulation. Section 3 deals with the model analysis to charcterise the optimal strategies and illustrates it with a numerical example. Finally, in Section 4, we conclude with some comments and briefly discuss some extensions to the model studied in this paper.

## 2. MODEL FORMULATION

In this section, we give the details of the model formulation.

### 2.1. Equipment Failures and Repairs

The equipment failure times are given by an exponential distribution with failure rate $\lambda$. Failed equipment is minimally repaired so that repaired equipment failure times are also exponentially distributed with failure rate $\lambda$. The time to repair is also exponentially distributed with repair rate $\mu$ (or equivalently, mean time for a repair is $1/\mu$). When the equipment is in working state, it generates a revenue $R$ per unit time and no revenue when it is in failed state. The useful life of the equipment is $L$ and the purchase price of a unit is $C_b$.

### 2.2. Repair Options

The agent offers the following two options to each customer.

OPTION $A_1$. (Service contract.) For a fixed price of $P$, the agent agrees to repair all failures over $[0, L)$ at no additional cost. If a failed unit is not returned in operational mode within a period $\tau$ subsequent to the failure, the agent incurs a penalty. Let $Y$ (a random variable) denote the time to return a failed unit back to operational state. The penalty incurred is $\alpha(Y - \tau)$ if $Y > \tau$ and zero, otherwise.

OPTION $A_2$. (No service contract.) In this case, whenever the unit fails, the customer gets it repaired at a cost of $C_s$. There is no penalty regarding the time taken to rectify the failed unit. Under this option, the total cost of repair over the life period $L$ is a random variable.

Let $M$ denote the number of customers serviced by the agent.

### 2.3. Customer's Decision Problem

Each customer has to choose between options $A_1$ and $A_2$ with the purchase of the equipment. The third option $A_0$ is simply not to purchase the equipment. $A_0$ is the optimal option if $P$ and $C_s$ are so large that the expected returns to the customer are negative rather than positive.

We assume that all $M$ customers are identical in their attitude to risk and that their optimal choice is based on maximising an expected utility function. We use the following utility function:

$$U(\omega) = \frac{1 - e^{-\beta\omega}}{\beta}, \tag{1}$$

where $U(\omega)$ is the utility associated with a wealth of $\omega$. The advantage of this utility function is that the initial wealth is of no importance. (See [2] for further discussion.) Note that $\beta = 0$ corresponds to the risk neutral case with $U(\omega) = \omega$.

For customer $j$ ($1 \le j \le M$), let the number of failures over $[0, L)$ be $N_j$. Let $X_{ji}$ denote the time to $i^{\text{th}}$ ($0 \le i \le N_j$) failure after $(i-1)^{\text{th}}$ repair. Let $\widetilde{X}_j$ denote the time for which the unit was in operational state at the end of its useful life subsequent to being restored to operational state after the last repair. Note, that it is zero if the unit is in failed state when it reaches the end of its useful life. Let $Y_{ji}$ ($0 \le i \le N_j$) denote the time taken to make the unit operational after the $i^{th}$ failure. This time includes the waiting time and the time to repair. Then, under the three options ($A_1$, $A_2$, and $A_0$, respectively), the returns (denote by $\omega(A_k)$, $0 \le k \le 2$) to the $j^{\text{th}}$ customer are as follows:

$$\omega(A_1) = R\left(\sum_{i=0}^{N_j} X_{ji} + \widetilde{X}_j\right) + \alpha\left(\sum_{i=0}^{N_j} \max\{0, (Y_{ji} - \tau)\}\right) - C_b - P, \tag{2}$$

$$\omega(A_2) = R\left(\sum_{i=0}^{N_j} X_{ji} + \widetilde{X}_j\right) - C_b - C_s N_j, \tag{3}$$

and finally, $\omega(A_0) = 0$. Note that $\omega(A_1)$ and $\omega(A_2)$ are random variables as failures occur in an uncertain manner.

Since all the customers are similar in their attitude to risk and since all units are statistically similar, the expected utility under action $A_k$ ($0 \le k \le 2$) is the same for all $M$ customers. Let $U(A_k; P, C_s, M)$ denote the expected utility to a customer when action $A_k$ ($0 \le k \le 2$) is selected. This is obtained using (1) with the appropriate $\omega(A_k)$ and carrying out the expectation. The optimal action $A*$ $[= A * (P, C_s, M)]$ is selected from the set $\{A_0, A_1, A_2\}$ to yield the maximum expected utility. Note that it is a function of $P$, $C_s$, and $M$, the agent's decision variables.

### 2.4. Agent's Decision Problem

The agent is assumed to be risk neutral. The profits derived from customer $j$ depends on the actions of the customers. Since all the customers are similar, they choose the same action. Let the agent's profit under action $A_k$ by the customer be denoted by $\pi(P, C_s, M; A_k)$, ($0 \le k \le 2$). It is easily seen that

$$\pi(P, C_s, M; A_1) = \sum_{j=1}^{M}\left[P - C_\tau N_j - \alpha\left(\sum_{i=0}^{N_j} \max\{0, (Y_{ji} - \tau)\}\right)\right], \tag{4}$$

$$\pi(P, C_s, M; A_2) = \sum_{j=1}^{m} [C_s - C_r] N_j, \tag{5}$$

and $\pi(P, C_s, M; A_0) = 0$, where $C_r$ is the cost of each repair.

The agent's optimal choice of $P$, $C_s$, and $M$ are obtained by maximising the expected profits, taking into account the optimal action $A*$ $(= A * (P, C_s, M))$ of the customers. In other words, the optimal actions are given by the solution of a Stackelberg game formulation with the agent being the leader and customers being the followers.

## 3. MODEL ANALYSIS

We make some simplifying assumptions so that the analysis is tractable. These are as follows.

(i) Failed units are repaired on a first-come, first-repair basis.

(ii) We use the steady state distribution for $Y_{ji}$ in our analysis. This is a valid assumption if $L$ is sufficiently large.

(iii) $E(Y_{ji}) \ll (1/\lambda)$. This implies that the mean total (waiting + repair) time is very small in relation to mean time to failure. This is a valid assumption for well-designed equipment (so that mean time to repair is small) and the number of customers $M$ is not too large (so that the mean waiting time is small). As a result, the total down time of equipment for each customer is negligible and can be ignored.

(iv) Both the agent and the customers have more complete information regarding the model parameters.

Note, in the last section, we discuss extensions which relax some of these assumptions.

### 3.1. Steady State Distribution for $Y_{ji}$

Note, that the formulation is identical to a Markovian queue with finite population ($M$) with a single server and first-come, first-served rule. When there are $K$ failed units, then the arrival rate of failed units is $\lambda_k = (M-K)\lambda$ for $0 \le K \le M$ and $= 0$ for $K > M$. The service (repair) rate is $\mu$ since there is only one server. We assume that $M\lambda < \mu$. If not, the queue builds up with time, and as a result the total time each failed equipment is in the system waiting to be repaired increases. We assume that the queue reaches steady state in a relatively short time so that one can use the steady state results. Let $F(y)$ and $f(y)$ denote the steady state distribution and density functions, respectively, for the random variable $Y_{ji}$. Expressions for these can be obtained using results from queueing theory. From Gross and Harris [3] or White *et al.* [4], we have

$$f(y) = \sum_{k=0}^{M-1} \widehat{P}_k \mu e^{-\mu y} \frac{(\mu y)^k}{k!}, \tag{6}$$

where $\widehat{P}_k$ $(0 \le k \le M-1)$ are given by

$$\widehat{P}_k = \frac{(M-k)P_k}{\sum\limits_{k=0}^{M-1} (M-k)P_k}, \tag{7}$$

with

$$P_k = \frac{(\lambda/\mu)^k \{M!/(M-k)!\}}{\sum\limits_{k=0}^{M-1} (\lambda/\mu)^k \{M!/(M-k)!\}}. \tag{8}$$

The expected value of $Y_{ji}$ is given by

$$E[Y_{ji}] = \sum_{k=0}^{M-1} \frac{\widehat{P}_k (k+1)}{\mu}. \tag{9}$$

### 3.2. Customer's Optimal Strategy

Because of Assumption (iii), we have

$$R\left(\sum_{i=0}^{N_j} X_{ji} + \widetilde{X}_j\right) \approx RL,$$

and $N_j$ is a renewal process with intensity function $\lambda$. In other words, $N_j$ is Poisson distributed with parameter $\lambda$.

For customer $j$, let $\widetilde{N}_j$ denote the number of times the agent incurs penalty. The expectation $E[U(A_1; P, C_s, M)]$ needs to be evaluated over $N_j$, $\widetilde{N}_j$, and the $Y_{ji}s$. We do this using a three stage conditional approach.

Let $E[U(A_1; P, C_s, M \mid \widetilde{N}_j, N_j)]$ denote the expectation over random variables $Y_{ji}s$ conditional on $\widetilde{N}_j$ and $N_j$. It is given by

$$E\Big[U\left(A_1; P, C_s, M \mid \widetilde{N}_j, N_j\right)\Big] = \left(\frac{1}{\beta}\right)\left[1-e^{-\beta(RL-C_b-P)}\right] \times \left(\int_\tau^\infty e^{-\beta\alpha(y-\tau)}\left[\frac{f(y)}{1-F(\tau)}\right] dy\right)^{\widetilde{N}_j}.$$

Note, that $\widetilde{N}_j$ conditional on $N_j$ is a binomially distributed random variable with parameter $F(\tau)$. Using this in the unconditioning process, we have

$$E[U(A_1; P, C_s, M \mid N_j)] = \left(\frac{1}{\beta}\right) - \psi\left\{\int_\tau^\infty e^{-\beta\alpha(y-\tau)} f(y)\, dy + F(\tau)\right\}^{N_j},$$

where $\psi = (1/\beta)e^{-\beta(RL-C_b-P)}$. From Assumption (iii), $N_j$ is Poisson distributed with parameter $\lambda$. Using this in the final unconditioning, we have

$$E[U(A_1; P, C_s, M)] = \left(\frac{1}{\beta}\right)\left\{1 - e^{-\beta(RL-C_b-P)+\lambda L\left[\int_\tau^\infty e^{-\beta\alpha(y-\tau)} f(y)\, dy + F(\tau)-1\right]}\right\}. \tag{10}$$

Using a similar approach, we have

$$E[U(A_2; P, C_s, M)] = \left(\frac{1}{\beta}\right)\left\{1 - e^{-\beta(RL-C_b)-\lambda L\left(1-e^{\beta C_s}\right)}\right\}. \tag{11}$$

Finally, note that $E[U(A_0; P, C_s, M)] = 0$.

For a given $(P, C_s, M)$, a comparison between the three expected utilities will indicate which action is the optimal one. For a fixed $M$, the optimal customer strategy is characterised by three regions $(\Omega_i,\ 0 \le i \le 2)$, in the $(P, C_s)$ plane as shown in Figure 1. In $\Omega_0$, $A* = A_0$; in $\Omega_1$, $A* = A_1$, and in $\Omega_2$, $A* = A_2$. The curve $(\Gamma)$ separating $\Omega_1$ and $\Omega_2$ is given by

$$\begin{aligned} P = \left(\frac{\lambda L}{\beta}\right)\Bigg\{e^{\beta C_s} - \sum_{k=0}^{M-1} \widehat{P}_k \Bigg[\mu^{k+1} e^{-\mu\tau} \sum_{j=0}^{k}\left\{\frac{\tau^{k-j}}{(k-j)!}\right\} \\ \times\left\{\left(\frac{1}{(\beta\alpha+\mu)^{j+1}}\right) - \left(\frac{1}{\mu^{j+1}}\right)\right\} + 1\Bigg]\Bigg\}, \end{aligned} \tag{12}$$

and $\bar{P}(M)$ and $\bar{C}_s(M)$ are given by

$$\begin{aligned} \bar{P}(M) = RL - C_b - \left(\frac{\lambda L}{\beta}\right)\sum_{k=0}^{M-1} \widehat{P}_k \Bigg[\mu^{k+1} e^{-\mu\tau} \sum_{j=0}^{k}\left\{\frac{\tau^{k-j}}{(k-j)!}\right\} \\ \times\left\{\left(\frac{1}{(\beta\alpha+\mu)^{j+1}}\right) - \left(\frac{1}{\mu^{j+1}}\right)\right\}\Bigg], \end{aligned} \tag{13}$$

$$\bar{C}_s(M) = \left(\frac{1}{\beta}\right)\ln\left\{1 + \frac{\beta(RL - C_b)}{(\lambda L)}\right\}. \tag{14}$$

As $M$ increases, the curve $\Gamma$ moves upwards. $\bar{P}(M)$ increases, as $M$ increases but $\bar{C}_s(M)$ does not change with $M$.

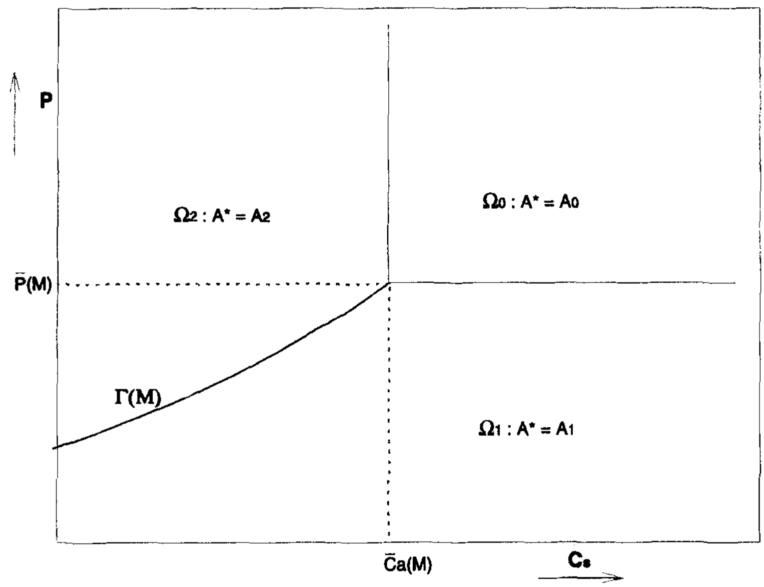


Figure 1. Characterization of customer's optimal actions for a given $M$.

### 3.3. Agent's Optimal Strategy

We derive the agent's optimal strategy as follows. Let $P*(M)$ and $C_s*(M)$ denote the optimal $P$ and $C_s$ which maximizes the agent's expected profit for a given $M$. Once this is done, $M*$, the optimal $M$ is obtained by varying $M$. Note that $M$ is constrained to the set of positive integers less than $\mu/\lambda$. The optimal $P$ and $C_s$ are given by $P*(M*)$ and $C_s*(M*)$.

For notational convenience, we omit the argument $M$ so that $P*(M)$ and $C_s*(M)$ will be simply referred to as $P*$ and $C_s*$.

It can be shown that for a given $M$, the agent must select either $P > \bar{P}$ and $C_s = \bar{C}_s$ or $P = \bar{P}$ and $C_s > \bar{C}_s$ to maximise the expected profit when customers choose the optimal strategy to maximise their utility function. In this case, the expected agent's profit is given by

$$E[\pi(P, C_s, M; A*)] = \begin{cases} M\left(\bar{P} - \lambda L\left[C_r + \int_\tau^\infty \alpha(y-\tau) f(y)\,dy\right]\right), & \text{when } C_s > \bar{C}_s,\ P = \bar{P}, \\ M\lambda L\left[\bar{C}_s - C_r\right], & \text{when } C_s = \bar{C}_s,\ P > \bar{P}, \\ 0, & \text{when } C_s > \bar{C}_s,\ P > \bar{P}. \end{cases} \quad (15)$$

When $P > \bar{P}$ and $C_s = \bar{C}_s(M)$, the customer's optimal choice $A*$ is $A_2$; when $P = \bar{P}(M)$ and $C_s > \bar{C}_s(M)$, then $A*$ is $A_1$, and finally, when $P > \bar{P}(M)$ and $C_s > \bar{C}_s(M)$, then $A*$ is $A_0$. In the first two cases, the customer's expected utility is zero and there is no consumer surplus. This implies that the agent, as a monopolist, extracts the maximum amount from the customer. Charging any more, i.e., $P > \bar{P}(M)$ and $C_s > \bar{C}_s(M)$, results in customers choosing $A_0$ instead of $A_1$ or $A_2$.

As a result, for a given $M$, the agent's optimal action is the choice between

(1) $P* > \bar{P}(M)$ and $C_s* = \bar{C}_s(M)$, and
(2) $P* = \bar{P}(M)$ and $C_s* > \bar{C}_s(M)$.

The one which yields a higher expected profit is the optimal choice. The optimal $M$ is determined by varying $M$ from 1 to the largest integer $\leq [\mu/\lambda]$. This can be obtained using an enumerative method.

### 3.4. Sensitivity Analysis

The two important model parameters are $\beta$ (the risk parameter) and $\lambda$ (the failure rate of the equipment). We discuss the effect of the variations of these parameters on the optimal solution.

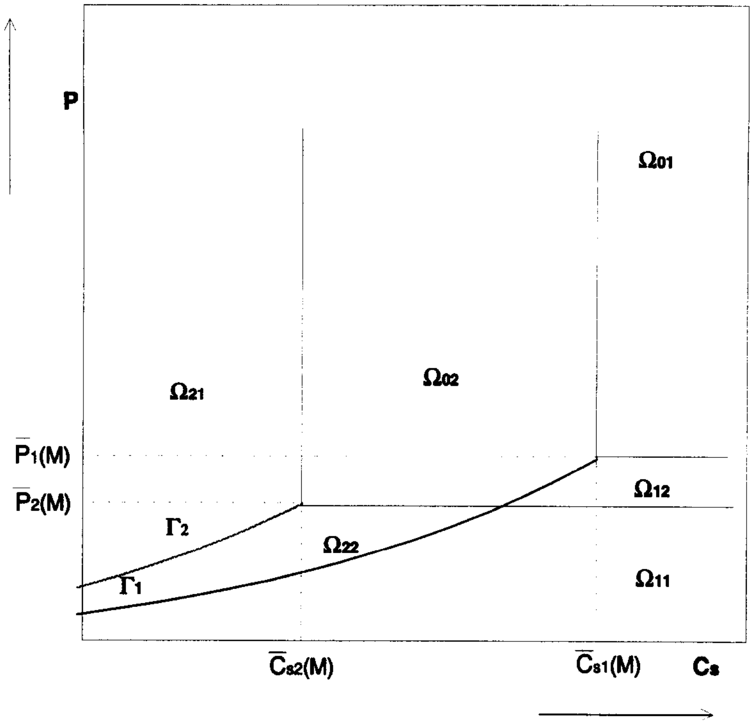


Figure 2. Effect of $\beta$ on the customer's optimal actions for a given $M$.

EFFECT OF $\beta$ VARIATIONS. It can be easily shown that $\frac{d\bar{P}(M)}{d\beta} < 0$ and $\frac{d\bar{C}_s(M)}{d\beta} < 0$. This implies that both $\bar{P}(M)$ and $\bar{C}_s(M)$ decrease as $\beta$ increases. Also, the curve $\Gamma$ moves upward as $\beta$ increases. As a result, with $\beta$ increasing, the horizontal line (corresponding to $P = \bar{P}(M)$) moves down and the vertical line (corresponding to $C_s = \bar{C}_s(M)$) moves to the left. Figure 2 shows the plots for two values of $\beta$ with $\beta_2 > \beta_1$.

Let $\bar{P}_i(M)$ and $\bar{C}_{si}(M)$ denote the $\bar{P}(M)$ and $\bar{C}_s(M)$ for $\beta_i$, $1 \le i \le 2$. For $0 < C_s < \bar{C}_{s2}(M)$, we see that for a fixed $C_s$, a more risk averse customer is willing to pay a higher price for service contract relative to a low risk customer. Similarly, for a fixed $P(> \bar{P}_1(M))$, a low risk customer is willing to buy the equipment and get each failure repaired individually as long as $C_s < \bar{C}_{s1}(M)$, whereas a high risk customer will opt for $A_0$ when $C_s > \bar{C}_{s2}(M)$. These results are as to be expected and agree intuitively with the anticipated behaviour of customers with increasing risk aversion.

When $\beta = 0$ corresponds to both customers and agent being risk neutral, the agent's optimal profit (for a given $M$) under options $A_1$ and $A_2$ are the same and is given by

$$E[\pi(P*, C_s*, M; A* = A_1)] = E[\pi(P*, C_s*, M; A* = A_2)] = M(RL - C_b - \lambda LC_r).$$

The waiting time to repair does not have any impact on the agent's profit. The reason for this is that the agent recovers the penalty costs through a higher value for $P*$, and the customers do not mind this as they get compensation through the penalty payments made by the agent. The optimal profit is a linear function of $M$ and $0 < M \le [\mu/\lambda]$.

EFFECT OF $\lambda$ VARIATIONS. It can be easily shown that $\frac{d\bar{P}(M)}{d\lambda} > 0$ and $\frac{d\bar{C}_s(M)}{d\lambda} < 0$. This implies that $\bar{P}(M)$ increases and $\bar{C}_s(M)$ decreases as $\lambda$ increases. Also, the curve $\Gamma$ moves upward as $\lambda$ increases. As a result, with $\lambda$ increasing, the horizontal line (corresponding to $P = \bar{P}(M)$) moves up and the vertical line (corresponding to $C_s = \bar{C}_s(M)$) moves to the left. The consequence of this is that the region $\Omega_1$ increases and $\Omega_2$ shrinks as $\lambda$ increases.

Also as $\lambda$ increases, the maximum number of customers that the agent can service decreases as $M < [\mu/\lambda]$. This makes intuitive sense as the number of times an equipment fails increases (in a probabilistic sense) as $\lambda$ increases.

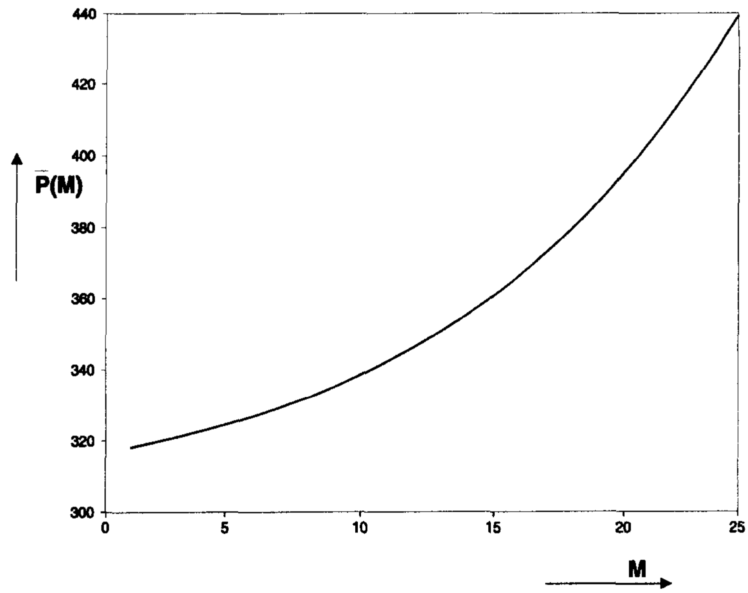


Figure 3. Plot of $\bar{P}(M)$ vs. $M$.

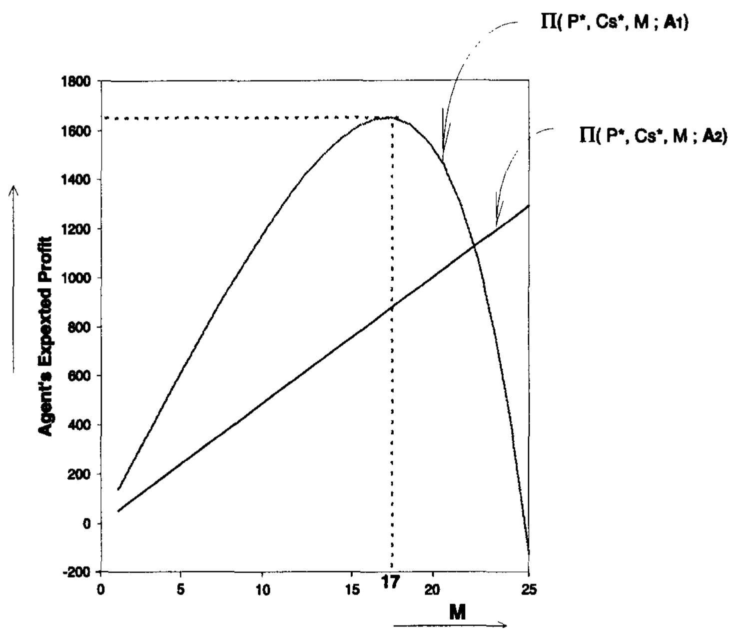


Figure 4. $E[\pi(P*, C_s*, M; A_i$ vs. $M$ for $i = 1, 2)]$.

### 3.5. Numerical Example

Consider the following nominal values for the model parameters: $\lambda = 0.0008$ (per hour), $\mu = 0.02$ (per hour), $\alpha = 0.06$ $((10)^3\$$ per hour), $\beta = 0.1$, $\tau = 70$ (hours), $C_b = 300((10)^3\$)$, $L = 40,000$ (hours), $C_r = 5(10)^3\$$, and $R = 0.015((10)^3\$$ per hour). These values imply that $M < 25[= \mu/\lambda]$.

As mentioned earlier, $\bar{C}_s(M)$ does not depend on $M$ and is $6.6139((10)^3\$)$. In contrast, $\bar{P}(M)$ depends on $M$. Figure 3 shows a plot of $\bar{P}(M)$ versus $M$ with $M$ varying from 1 to 25. Note that $\bar{P}(M)$ increases with $M$. The reason for this is as follows. $M$ increasing results in longer waiting time for repair, and hence, greater expected penalty cost. As a result, the service contract price must increase.

Figure 4 shows the plot of $E[\pi(P*, C_s*, M; A*)]$ as a function of $M$. For each value of $M$, we consider $P*$ and $C_s*$ combinations which result in $A*$ being $A_1$ or $A_2$. For $1 < M < 21$, the agent

Table 1. $\bar{P}(M)$ vs. $M$ for $\beta = 0.1$ and 0.2.

| $\beta$ | $M$ | | | | | |
|---|---|---|---|---|---|---|
| | 1 | 5 | 10 | 15 | 20 | 25 |
| 0.1 | 318.210 | 325.171 | 338.197 | 358.732 | 390.959 | 439.190 |
| 0.2 | 314.795 | 320.006 | 329.346 | 343.204 | 363.294 | 390.362 |

Table 2. $A*$, $M*$, and $E[\pi(P*, C_s*, M*; A*)]$ vs. $\beta$.

| | $\beta$ | | | | |
|---|---|---|---|---|---|
| | 0.0 | 0.05 | 0.10 | 0.15 | 0.20 |
| $A*$ | $A_1$ or $A_2$ | $A_1(M \le 21)$ $A_2(M > 21)$ | $A_1(M \le 21)$ $A_2(M > 21)$ | $A_1(M \le 22)$ $A_2(M > 22)$ | $A_1(M \le 22)$ $A_2(M > 22)$ |
| $M*$ | 24 | 19 | 17 | 16 | 15 |
| $E\pi$ | 3360.000 $(10^3)$ | 1976.808 $(10^3)$ | 1650.205 $(10^3)$ | 1484.578 $(10^3)$ | 1383.599 $(10^3)$ |

Table 3. $A*$, $M*$, and $E[\pi(P*, C_s*, M*; A*)]$ vs. $\lambda$.

| | $\lambda$ | | | |
|---|---|---|---|---|
| | 0.0002 | 0.0004 | 0.0008 | 0.0010 |
| $A*$ | $A_1(M \le 106)$ $A_2(M > 106)$ | $A_1(M \le 50)$ $A_2(M > 50)$ | $A_1(M \le 21)$ $A_2(M > 21)$ | $A_1(M \le 16)$ $A_2(M > 16)$ |
| $M*$ | 85 | 40 | 17 | 12 |
| $E[\pi]$ | 18385.251 $(10^3)$ | 6834.655 $(10^3)$ | 1650.205 $(10^3)$ | 761.349 $(10^3)$ |

must choose $P*$ and $C_s*$ so that the customers optimal actions are $A_1$. As can be seen from the figure, $M*$, the optimal $M$, is 17. The optimal pricing structure is given by $C_s* > 6.6139((10)^3)$ and $P* = 369.948((10)^3)$. The optimal expected profits to the agent is \$1650.205$((10)^3)$.

EFFECT OF $\beta$ VARIATIONS. When $\beta$ increases to 0.2, $\bar{C}_s(M)$ decreases to 5.2802 and $\bar{P}(M)$ values (for different $M$) are shown in Table 1. $\bar{P}(M)$ decreases as $\beta$ increases, as to be expected.

The optimal customer's choice $A*$ (in response to the optimal pricing) and $M*$ and the optimal expected profit $E[\pi(P*, C_s*, M*, A*)]$ (denoted by $E[\pi]$ in the table) for the agent for different values of $\beta$ are shown in Table 2.

As can be seen, $M*$ and the optimal expected profit $E[\pi(P*, C_s*, M*; A*)]$ decrease as $\beta$ increases. This implies that as customers become more risk averse, the optimal strategy for the agent is to reduce the number of customers in order to maximize the expected profit. Note that the optimal pricing results in customers always choosing option $A_1$.

EFFECT OF $\lambda$ VARIATIONS. The optimal customer's choice $A*$ (in response to the optimal pricing) and $M*$ and the optimal expected profit $E[\pi(P*, C_s*, M*; A*)]$ for the agent for different values of $\lambda$ are shown in Table 3.

As can be seen, $M*$ and the optimal expected profit $E[\pi(P*, C_s*, M*; A*)]$ decrease as $\lambda$ increases. This implies that as the product becomes more unreliable, the optimal strategy for the agent is again to reduce the number of customers in order to maximize the expected profit. Again, the optimal pricing results in customers always choosing option $A_1$.

## 4. COMMENTS AND SOME EXTENSIONS

In the model studied in this paper, as $M$ increases, the mean waiting time for failed items also increases. For large $M$, Assumption (iii) is no longer valid. In this case,

$$R\left(\sum_{i=0}^{N_j} X_{ji} + \tilde{X}_j\right) \approx R\left(L - \sum_{i=0}^{N_j} Y_{ji}\right),$$

and $N_j$ is an alternating renewal process with $X_{ji}$ exponentially distributed and $Y_{ji}$ distributed according to $F(y)$. (The steady state distribution for $F(y)$ was given in Section 3.1 and is valid only for large $L$.) This implies that the analysis is more complex and analytically intractable.

The model assumes that the failure rate is constant. This allowed the use of well-known results from queueing theory in the analysis. When the equipment has an increasing failure rate, the analysis becomes analytically intractable, and one would need to use simulation approaches to determine the optimal strategies. In this case, the failures over the life are influenced by other factors such as preventive maintenance and the type of corrective maintenance used.

The model can be extended in several ways. We list a few of these.

(i) The agent can repair more than one piece of failed equipment at any given time (i.e., the agent employs more than one repairman).

(ii) There is more than one type of service contract and these differ in their price and penalty clause.

(iii) The customers are not identical—in other words, the customer population is heterogeneous as opposed to being homogeneous and differ in their attitude to risk.

(iv) The customers do not know the true failure rate of the equipment. In this case, some might view the equipment as more reliable and others as less reliable.

(v) The case of two or more agents so that competition between agents becomes an important variable.

Some of these extensions are currently being investigated by the authors.

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