• Sea $f: [0,\infty) \to \mathbb{R}$ función periódica con período $T>0$.
Es decir $\forall t>0$. $f(t) = f(t+T)$. Demuestre que $\mathcal{L}[f](s) = \dfrac{\int_0^T e^{-s\xi} f(\xi)\,d\xi}{1-e^{-sT}}$

Solución

$$\mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^T e^{-st} f(t)\,dt + \int_T^{2T} e^{-st} f(t)\,dt + \int_{2T}^{3T} e^{-st} f(t)\,dt + \dots$$

$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT}\int_0^T e^{-st} f(t)\,dt + e^{-s2T}\int_0^T e^{-st} f(t)\,dt + \dots$$

cambio de variable en cada integral y el hecho de que $\forall k \in \mathbb{N}$: $f(t+kT) = f(t)$.

$$= \int_0^T e^{-st} f(t)\,dt \cdot \left\{ \sum_{k=0}^{\infty} (e^{-sT})^k \right\}$$

$$= \frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}.$$ Transformada de una función periódica

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• (Función gama)

$\forall x>0$ definimos $\Gamma(x) := \int_0^\infty e^{-t} t^{x-1}\,dt$

1) Verifique que la integral converge $\forall x>0$.

Solución:

si $x=1$, $\Gamma(1) = \int_0^\infty e^{-t}\,dt < \infty$ ✓

si $x>1 \Rightarrow \exists T>0\ \exists \alpha<1\ \forall t>T: t^{x-1} < e^{\alpha t}$ (eventualmente todo polinomio se acota por una función exponencial).

$$\Rightarrow \Gamma(x) = \int_0^T e^{-t} t^{x-1}\,dt + \int_T^\infty e^{-t} t^{x-1}\,dt$$

$$\leq \text{cte}(T,x) + \int_T^\infty \underbrace{e^{-t} e^{\alpha t}}_{e^{\overbrace{(\alpha-1)}^{<0} t}}\,dt < \infty.$$

Si $0<x<1 \Rightarrow 0<1-x<1$

$$\Gamma(x)=\int_0^\infty \frac{e^{-t}}{t^{1-x}}dt \leq \int_0^\infty \frac{dt}{t^{1-x}} < \infty.$$

2) Pd: $\Gamma(x+1)=x\Gamma(x)$.

(integración por partes)

$$\Gamma(x+1)=\int_0^\infty e^{-t}t^{x+1-1}dt=\int_0^\infty e^{-t}t^{x}dt = -t^x e^{-t}\Big|_{t=0}^{t=\infty} - \int_0^\infty -e^{-t}x\,t^{x-1}dt$$

$$\begin{pmatrix} u=t^x \\ dv=e^{-t}dt \\ du=x\,t^{x-1}dt \\ v=-e^{-t}\end{pmatrix} \qquad = x\Gamma(x).$$

3). Pd: $\Gamma(n+1)=n!$, $n\in\mathbb{N}$.

$$\Gamma(n+1)=n\Gamma(n)=n(n-1)\Gamma(n-1)=\ldots=n(n-1)(n-2)\ldots 2\cdot 1 \qquad \text{Obs: } \Gamma(1)=1,\ \Gamma(0)=1$$

4) Sea $\alpha>-1$. Pd: $L[t^\alpha](s)=\dfrac{\Gamma(\alpha+1)}{s^{\alpha+1}}$.

Solución: $$L[t^\alpha](s)=\int_0^\infty e^{-su}u^\alpha du = \int_0^\infty e^{-t}\left(\frac{t}{s}\right)^\alpha\frac{dt}{s}=\int_0^\infty e^{-t}\,\frac{t^{(\alpha+1)-1}}{s^{\alpha+1}}dt$$

$$\begin{pmatrix} su=t \\ s\,du=dt\end{pmatrix}\ s>0 \qquad =\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}.$$

• (Transformada de Mellin).

$\forall s>0\ \forall y\in\mathcal{C}([0,1])$ definimos (cuando el límite existe) la Transformada de Mellin, por:

$$M[y](s)=\int_0^1 x^{s-1}y(x)dx.$$

1). Pd: $M[1](s)=\frac{1}{s}$

$M[x^\alpha y](s)=M[y](s+\alpha)$, $\forall \alpha\in\mathbb{R}$.

(s>0)

Solución: $$M[1](s)=\int_0^1 x^{s-1}\,1\,dx=\frac{1}{s}x^s\Big|_0^1=\frac{1}{s}\,1=\frac{1}{s}$$

$$M[x^\alpha y](s)=\int_0^1 x^{s-1}x^\alpha y\,dx=\int_0^1 x^{(s+\alpha)-1}y\,dx=M[y](s+\alpha).$$

2) si $\forall s>0$; $y \in \mathcal{C}([0,1])$ es tal que $\lim_{x \to 0^+} x^s y(x) = 0$ (*)

~~(esto dice que y decrece a 0 en infinito más fuerte que cualquier polinomio!)~~.

Pd: $M[xy'] = -sM[y] + y(1)$

(suponga y suficientemente diferenciable).

Solución. $M[xy'](s) = \int_0^1 x^{s-1} x y' dx = \int_0^1 x^s y' dx = x^s y \Big|_0^1 - \int_0^1 y s x^s dx$

$\overset{(*)}{=} y(1) - sM[y](s)$.

3) Definamos $z(x) = \int_x^1 \frac{y(u)}{u} du$.

Pd: $M[z](s) = \frac{1}{s} M[y](s)$.

Solución: $z(x) = -\int_1^x \frac{y(u)}{u} du$

$z'(x) = -\frac{y(x)}{x} \Rightarrow y(x) = -xz'(x)$

$\Rightarrow M[y](s) = -M[xz'](s) = sM[z](s) - \cancelto{0}{z(1)} = sM[z]$.

4) Resuelva la EDO:
$$\boxed{\begin{gathered} x(xy')' + 2xy' = 1 \\ y(1) = 0 \\ y'(1) = 0 \end{gathered}}$$
(asuma que $M[f] = M[g] \Rightarrow f = g$)

Solución. $M[x(xy')'] + 2M[xy'] = M[1]$ (Obs: $M[\cdot]$ es trivialmente lineal.)

$-sM[xy'](s) + \cancelto{0}{xy'\Big|_{x=1}} + 2(-sM[y](s) + \cancelto{0}{y(1)}) = \frac{1}{s}$

$-sM[xy'](s) - 2sM[y](s) = \frac{1}{s}$

$$-s\left\{-s\,M[y](s) + \cancel{y(1)}\right\} - 2s\,M[y](s) = \frac{1}{s}$$

$$\Longrightarrow s^2 M[y](s) - 2s\,M[y](s) = \frac{1}{s}.$$

$$\therefore M[y](s) = \frac{1}{s}\,\frac{1}{s}\cdot\frac{1}{(s-2)} = \frac{1}{s}\,\frac{1}{s}\,M(x^{-2})(s) \quad , \text{ pues } M[x^{-2}\cdot 1](s) = M[1](s-1).$$

$$= \frac{1}{s}\,M\left[\int_x^1 \frac{u^{-2}}{u}\,du\right](s)$$

$$= \frac{1}{s}\,M\left[\frac{x^{-2}}{2} - \frac{1}{2}\right](s) = M\left[\int_x^1 \left(\frac{u^{-2}}{2} - \frac{1}{2}\right)\frac{1}{u}\,du\right](s)$$

$$\Longrightarrow y(x) = \int_x^1 \left(u^{-2}/2 - 1/2\right)\frac{1}{u}\,du.$$

- Sea f continua por pedazos y de orden exponencial, además $f(0)=0$.
y es solución de la ecuación integro-diferencial.

$$(1) \qquad f'(t) = \operatorname{sen}(t) + \int_0^t f(t-\tau)\cos\tau\,d\tau.$$

Determine $L[f]$ y f.

Solución: $f'(t) = \operatorname{sen} t + (\cos * f)(t) \quad / L$

$$s\,L[f](s) - f(0) = L[\operatorname{sen} t](s) + L[\cos * f](s)$$

$$L[f](s) = \frac{1}{s}\,L[\operatorname{sen} t](s) + \frac{1}{s}\,L[\cos t](s)\,L[f](s)$$

$$\Longrightarrow L[f](s) = \frac{\frac{1}{s}\,L[\operatorname{sen} t](s)}{1 - \frac{1}{s}\,L[\cos t](s)} = \frac{1}{s^3}$$

$$\Longrightarrow \quad f(x) = x^2/2.$$

• Sea $a>0$, $n\in\mathbb{N}$.

$L_n := \mathcal{L}[\operatorname{sen}^n(at)](s)$.

1). L_1, L_2 ?

$L_1 = \mathcal{L}[\operatorname{sen}(at)](s) = \dfrac{a}{s^2+a^2}$

$L_2 = \mathcal{L}[\operatorname{sen}^2(at)](s) = \mathcal{L}[\frac{1}{2}(1-\cos(2at))](s) = \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4a^2}\right)$

obs: $\operatorname{sen}^2(\beta) = \frac{1}{2}(1-\cos(2\beta))$.

$\operatorname{sen}^2(\beta)+\cos^2(\beta)=1$.

2) Pd: $L_n = L_{n-2} - \mathcal{L}[\operatorname{sen}^{n-2}(at)\cos^2(at)]$ $\quad \forall n\geq 3$.

Sol: $L_n = \mathcal{L}[\operatorname{sen}^n(at)](s) = \mathcal{L}[\operatorname{sen}^{n-2}(at)\operatorname{sen}^2(at)](s)$

$= \mathcal{L}[\operatorname{sen}^{n-2}(at)(1-\cos^2(at))](s)$

$= L_{n-2} - \mathcal{L}[\operatorname{sen}^{n-2}(at)\cos^2(at)](s)$.

3) Pd. $\mathcal{L}[\operatorname{sen}^{n-2}(at)\cos^2(at)](s) = \left[\dfrac{s^2}{a^2 n(n-1)} + \dfrac{1}{n-1}\right] L_n$

Sol. obs. $(\operatorname{sen}^{n-1}(at))' = (n-1)\operatorname{sen}^{n-2}(at)\cos(at)a$

$(\operatorname{sen}^n(at))' = an\cos(at)\operatorname{sen}^{n-1}(at)$.

$\mathcal{L}[\operatorname{sen}^{n-2}(at)\cos^2(at)] = \int_0^\infty e^{-st}\operatorname{sen}^{n-2}(at)\cos(at)\cos(at)dt$

$= \dfrac{1}{a(n-1)}\int_0^\infty e^{-st}(\operatorname{sen}^{n-1}(at))'\cos(at)dt$

integración por partes.

$u = e^{-st}\cos(at)$

$dv = (\operatorname{sen}^{n-1}(at))'dt$

$v = \operatorname{sen}^{n-1}(at)$

$du = (-se^{-st}\cos(at) - a e^{-st}\operatorname{sen}(at))dt$

$= \dfrac{1}{a(n-1)}\left\{ e^{-st}\cos(at)\operatorname{sen}^{n-1}(at)\Big|_0^\infty - \int_0^\infty \operatorname{sen}^{n-1}(at)(-se^{-st}\cos(at) - a e^{-st}\operatorname{sen}(at))dt\right.$

$$\Rightarrow \mathcal{L}[\operatorname{sen}^{n-2}(at)\cos^2(at)] = \frac{1}{a(n-1)} s\mathcal{L}[\operatorname{sen}^{n-1}(at)\cos(at)] + \frac{1}{(n-1)} L_n$$

. Veamos el término: $\mathcal{L}[\operatorname{sen}^{n-1}(at)\cos(at)](s) = \int_0^\infty e^{-st}\operatorname{sen}^{n-1}(at)\cos(at)dt$

$$= \frac{1}{an}\int_0^\infty e^{-st}(\operatorname{sen}^n(at))' dt = \frac{1}{an}\left\{ e^{-st}\operatorname{sen}^n(at)\Big|_0^\infty - -s\int_0^\infty \operatorname{sen}^n(at)e^{-st}dt \right\}$$

$$= \frac{s}{an} L_n$$

$$\therefore \mathcal{L}[\operatorname{sen}^{n-2}(at)\cos^2(at)] = \frac{1}{a(n-1)} s\cdot\frac{s}{an} L_n + \frac{1}{(n-1)} L_n = \left(\frac{s^2}{a^2n(n-1)} + \frac{1}{(n+1)}\right)L_n.$$

4). $L_n = L_{n-2} - \left(\frac{s^2}{a^2n(n-1)} + \frac{1}{(n-1)}\right)L_n$

$$\Rightarrow L_n = \frac{a^2n(n-1)}{s^2+a^2n^2}\cdot L_{n-2}$$

(las formulas explícitas quedan propuestas)

5). $L_n = \frac{a^2n(n-1)}{s^2+a^2n^2}\cdot L_{n-2} = (n-1)an\left(\frac{an}{s^2+a^2n^2}\right)\cdot\mathcal{L}[\operatorname{sen}^{n-2}(at)](s)$

$$= an(n-1)\ \mathcal{L}[\operatorname{sen}((an)t)](s)\ \mathcal{L}[\operatorname{sen}^{n-2}(at)](s)$$

$$\Rightarrow \boxed{\operatorname{sen}^n(at) = an(n-1)\int_0^t \operatorname{sen}(an(t-\sigma))\operatorname{sen}^{n-2}(a\sigma)d\sigma}$$

pues: $\mathcal{L}[f]\mathcal{L}[g] = \mathcal{L}[f*g]$

$$\Rightarrow \mathcal{L}^{-1}(\mathcal{L}[f]\mathcal{L}[g]) = f*g.$$

- $\boxed{\begin{array}{c} ty'' + (t-1)y' + y = 0 \\ y(0)=0. \end{array}}$ resuelva:

Obs: $0y''(0) - 1y'(0) + y(0) = 0 \Longrightarrow y'(0) = 0.$

$$\mathcal{L}[ty''](s) + \mathcal{L}[ty'](s) - \mathcal{L}[y'](s) + \mathcal{L}[y](s) = 0.$$

$$-\frac{d}{ds}\mathcal{L}[y''](s) - \frac{d}{ds}\mathcal{L}[y'](s) - \mathcal{L}[y'](s) + \mathcal{L}[y] = 0.$$

$$\Longrightarrow -\frac{d}{ds}\left\{ s^2 L(s) - s\,y(0) - y'(0) \right\} - \frac{d}{ds}\left\{ sL(s) - y(0) \right\}$$

$$-\left\{ sL(s) - y(0) \right\} + L(s) = 0.$$

$$\longrightarrow L'(s)\,(s^2+s) + 3sL(s) = 0$$

$$\longrightarrow L(s) = cte\,\frac{1}{(1+s)^3}$$ (se resuelve a separación de variables)

$$\longrightarrow y(t) = cte\; e^{-t}\,\frac{t^2}{2}.$$

- $\mathcal{L}^{-1}\left[\operatorname{Arctg}\left(\frac{3}{2+s}\right) \right](t)$

Obs: $\frac{d}{ds}\mathcal{L}[f](s) = -\mathcal{L}[tf](s)$

$$\longrightarrow f(t) = -\frac{1}{t}\mathcal{L}^{-1}\left\{ \frac{d}{ds}\mathcal{L}[f](s) \right\}(t)$$

notar que $\frac{d}{ds}\underbrace{\operatorname{Arctg}\left(\frac{3}{2+s}\right)}_{\mathcal{L}[f](s)} = \frac{(2+s)^2}{9+(2+s)^2}\cdot\frac{-3}{(2+s)^2}$

$$= -\frac{3}{9+(s+2)^2}$$

$$\Longrightarrow tf(t) = \mathcal{L}^{-1}\left[\frac{3}{(s+2)^2+9}\right](t) = e^{-2t}\operatorname{sen}(3t)$$

$$\therefore f(t) = \frac{1}{t}\cdot e^{-2t}\operatorname{sen}(3t)$$